ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Biological Modeling of Neural Networks:

Week 13 – Membrane potential densities and Fokker-Planck

Wulfram Gerstner

EPFL, Lausanne, Switzerland

13.1 Review: Integrate-and-fire
- stochastic spike arrival

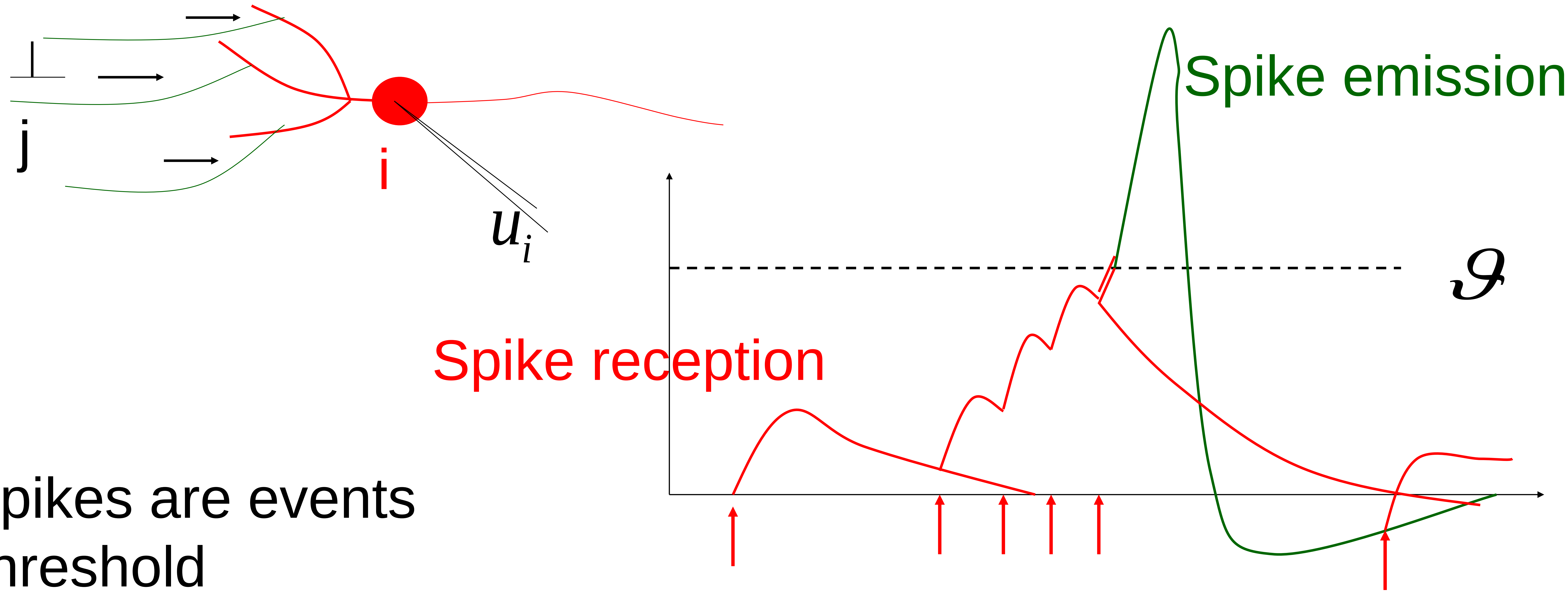
13.2 Density of membrane potential
- Continuity equation

13.3 Flux
- jump flux
- drift flux

13.4. Fokker –Planck Equation
- free solution

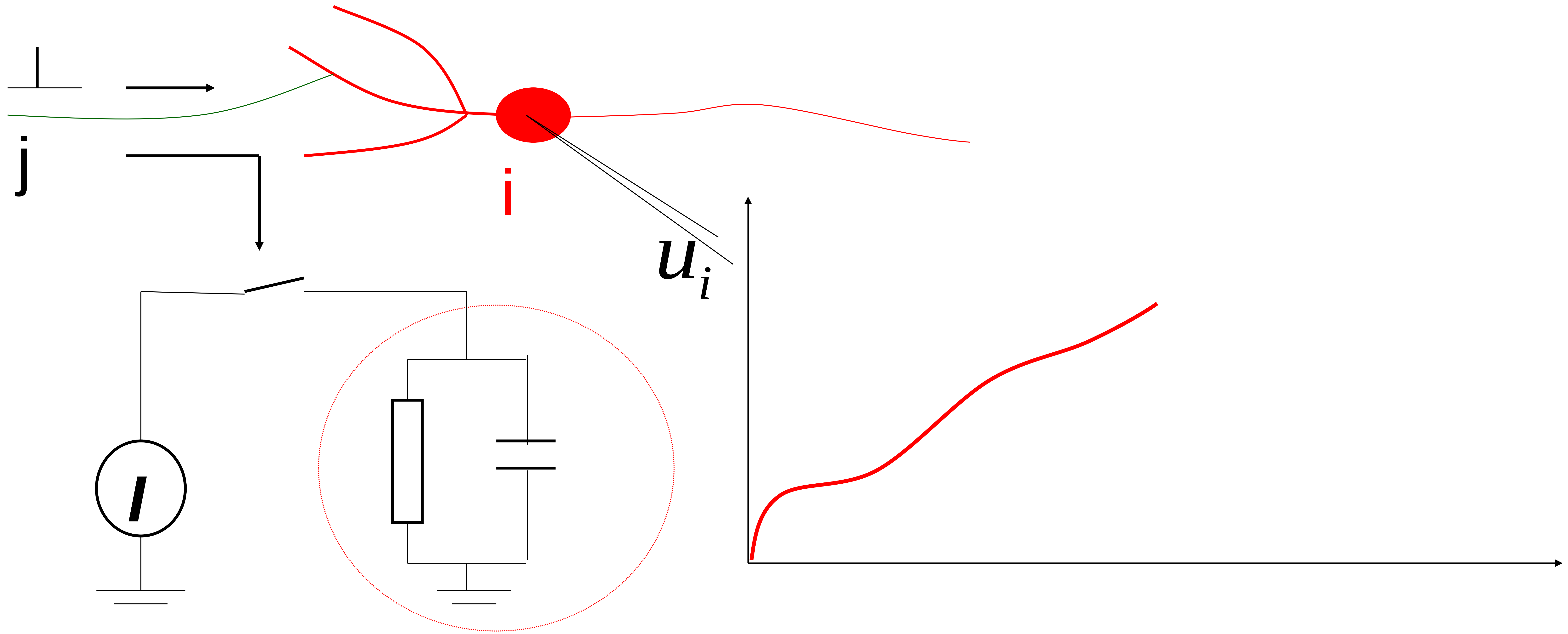
13.5. Threshold and reset
- time dependent activity
- network states

Week 13-part 1: Review: integrate-and-fire-type models



-spikes are events
-threshold
-spike/reset/refractoriness

Week 13-part 1: Review: leaky integrate-and-fire model



$$\tau \cdot \frac{d}{dt} u = -(u - u_{eq}) + RI(t)$$

If firing: $u \rightarrow u_{reset}$

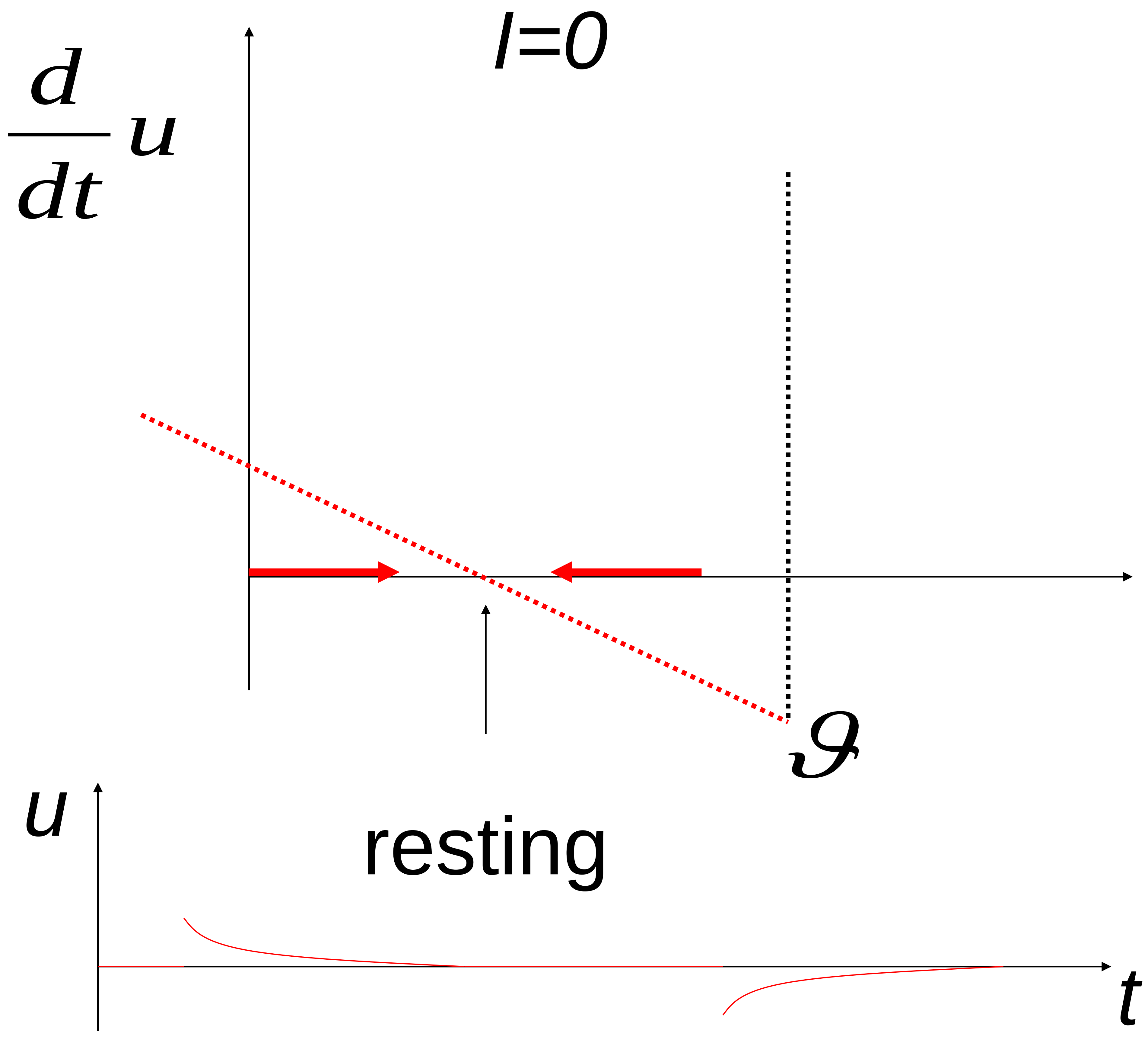
$$\tau \cdot \frac{d}{dt} V = -V + RI(t); \quad V = (u - u_{eq})$$

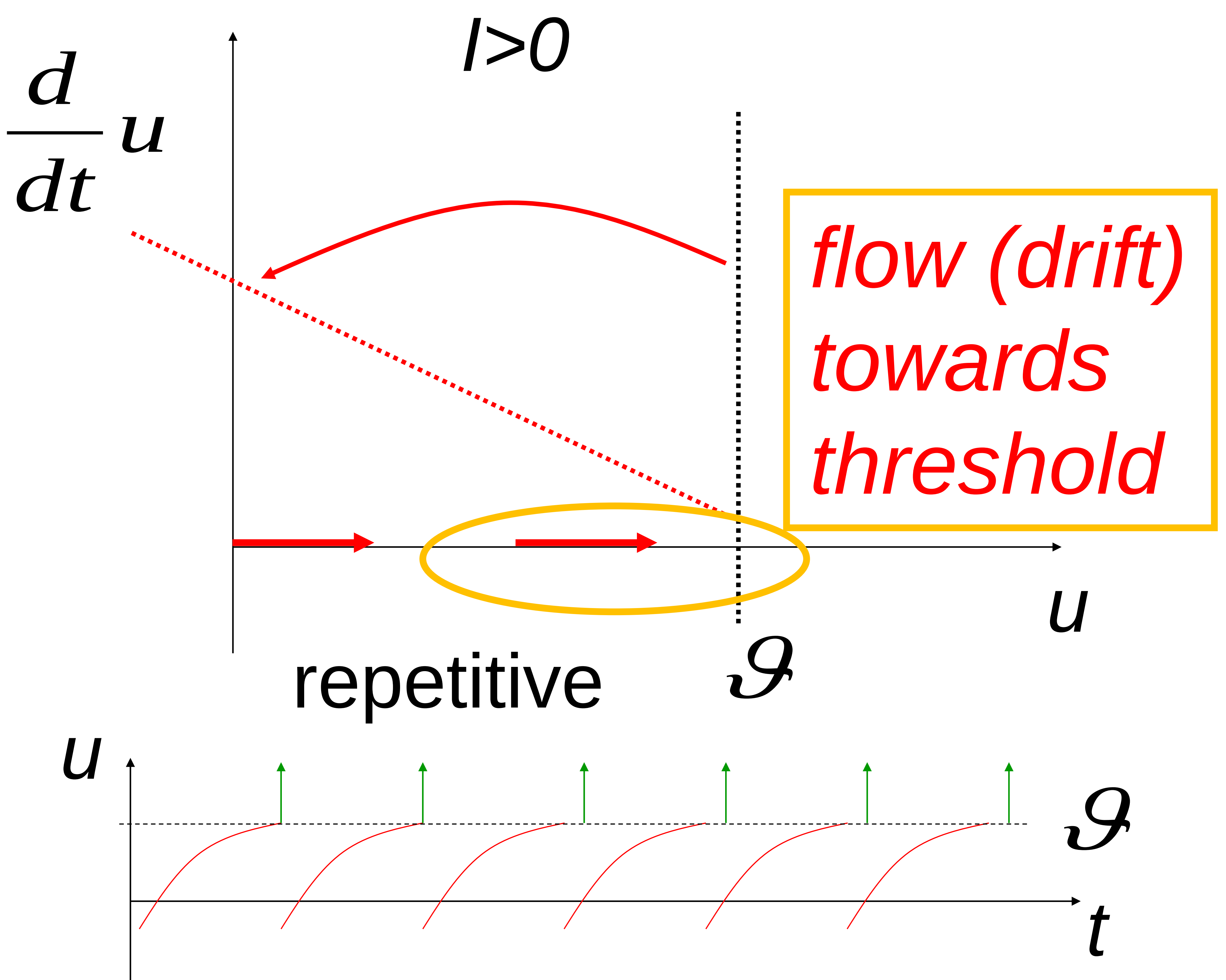
Week 13-part 1: Review: leaky integrate-and-fire model

$$\tau \cdot \frac{d}{dt}u = -(u - u_{eq}) + RI(t)$$

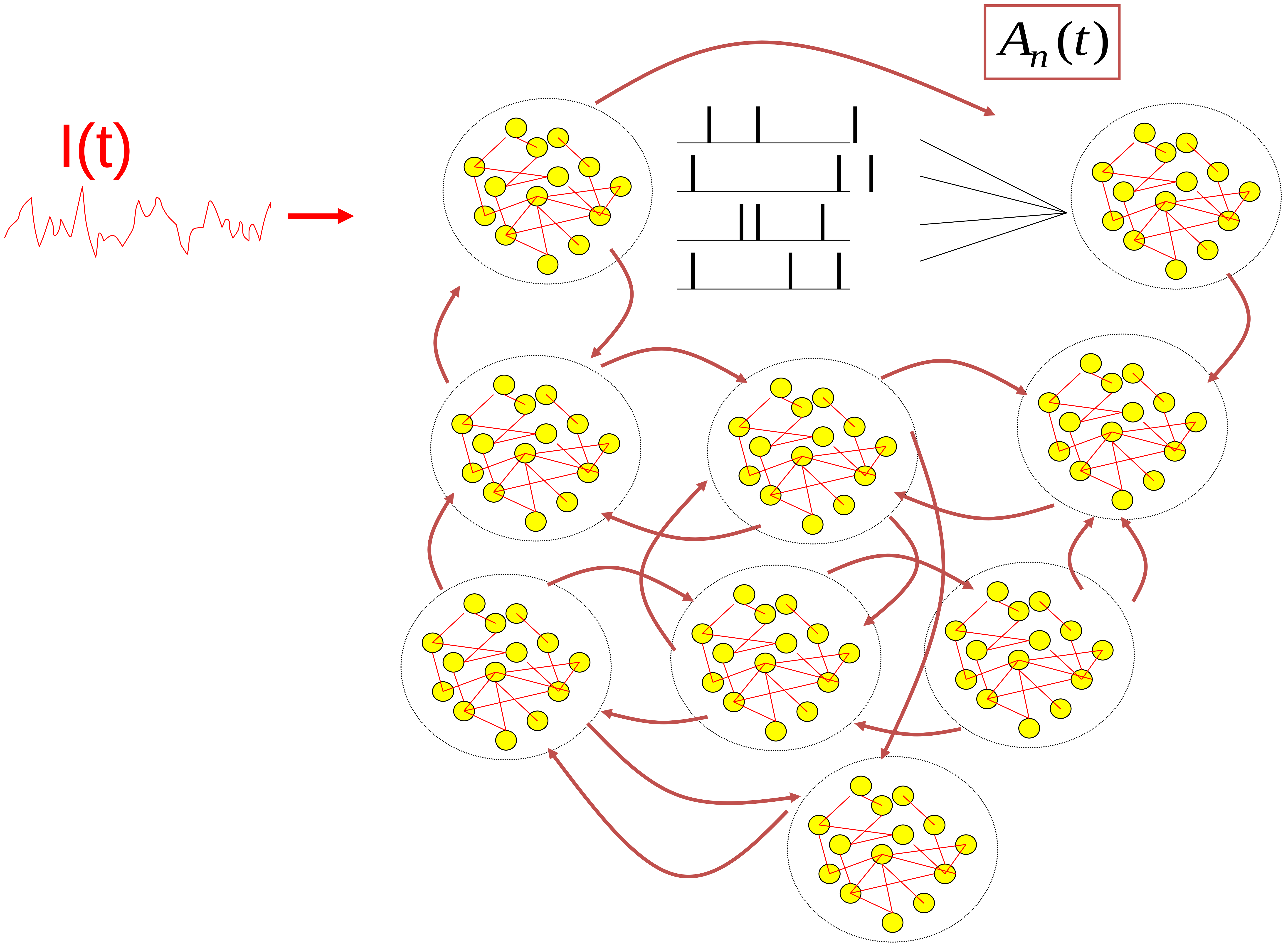
LIF

If firing: $u \rightarrow u_{reset}$

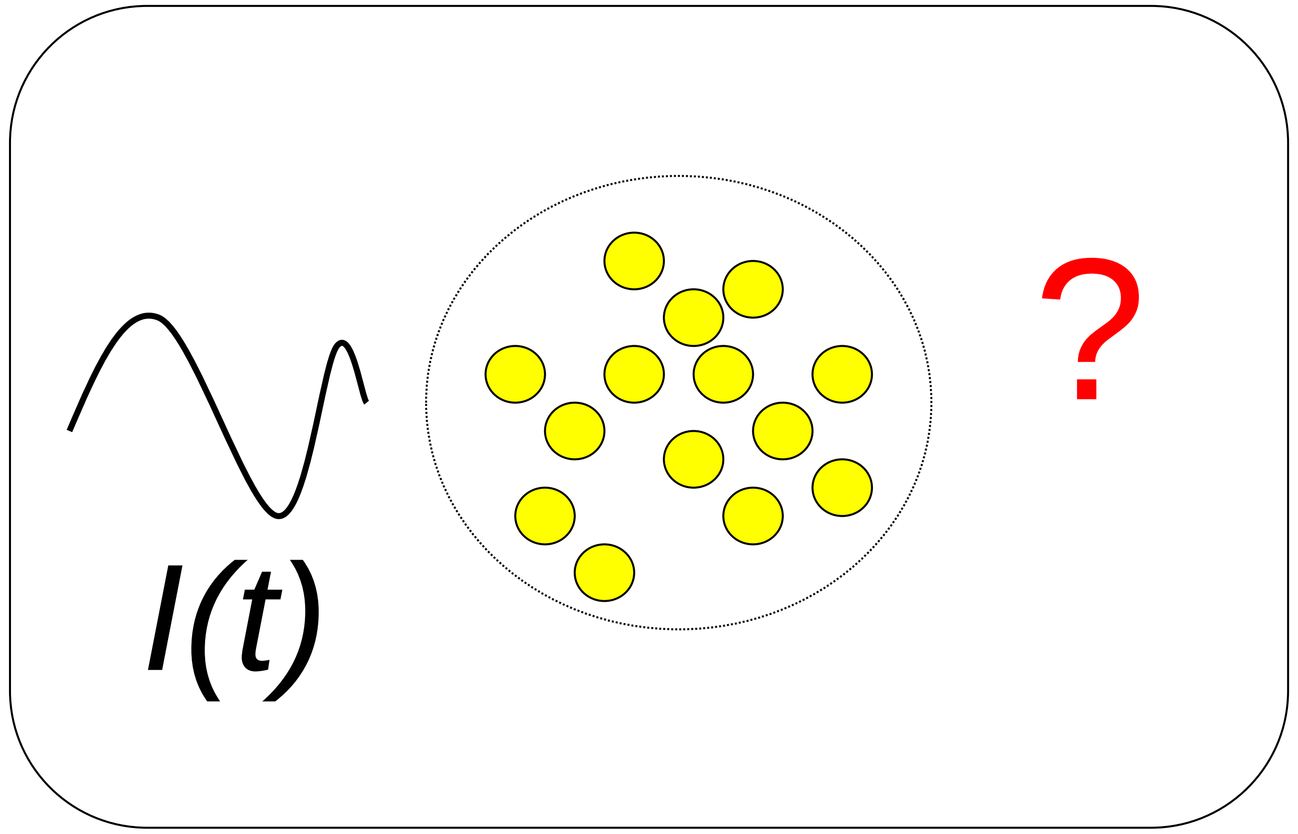


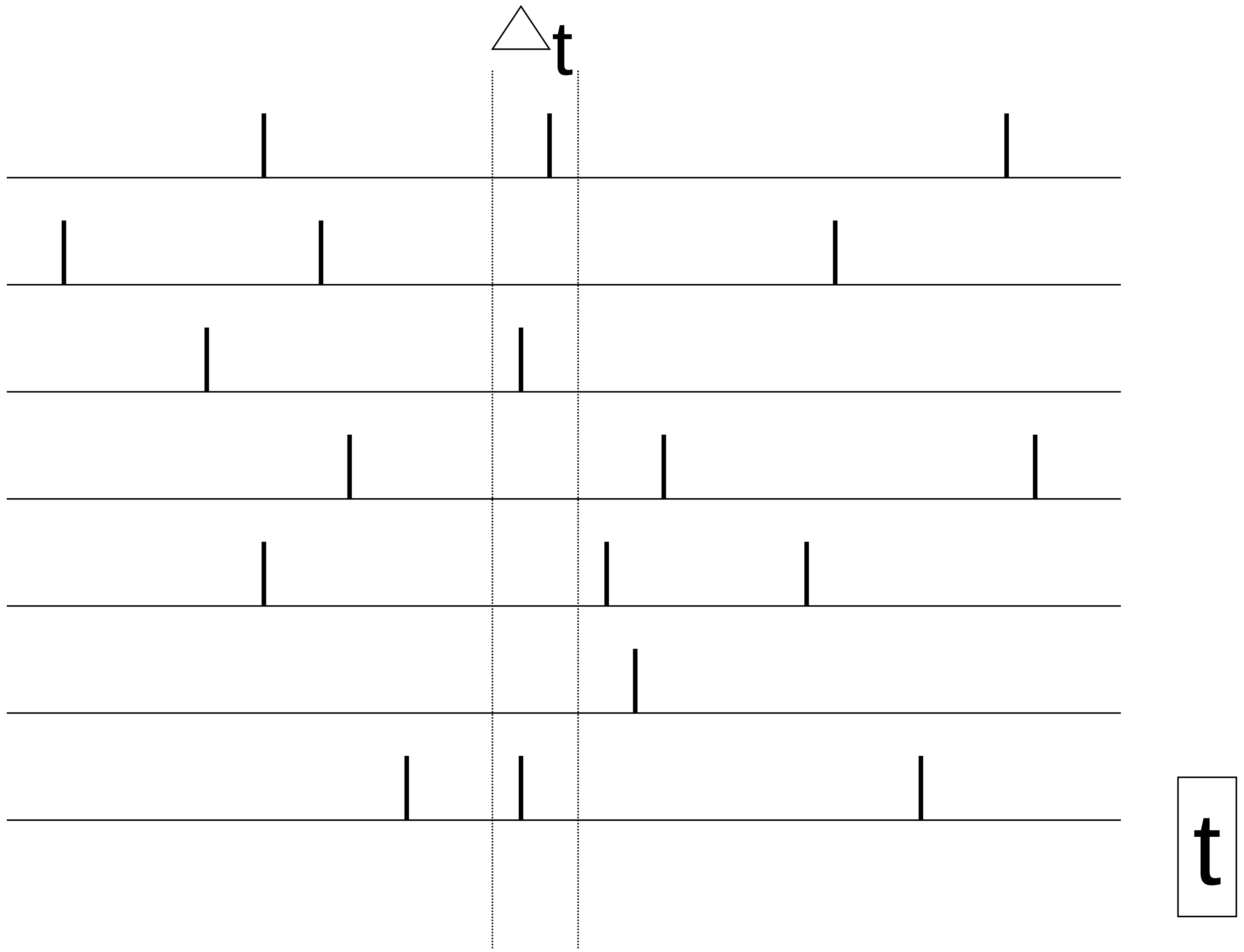


Week 13-part 1: Review: microscopic vs. macroscopic



Week 13-part 1: Review: homogeneous population





Homogeneous network:

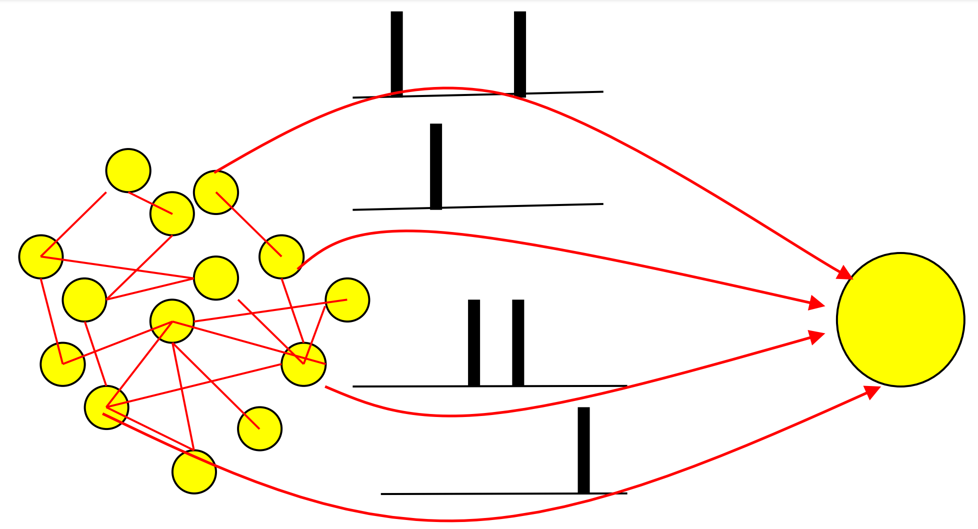
-each neuron receives input from k neurons in network

-each neuron receives the same (mean) external input

population activity

$$A(t) = \frac{n(t;t+\Delta t)}{N\Delta t}$$

Week 13-part 1: Review: diffusive noise/stochastic spike arrival



Stochastic spike arrival:

excitation, total rate R_e

inhibition, total rate R_i

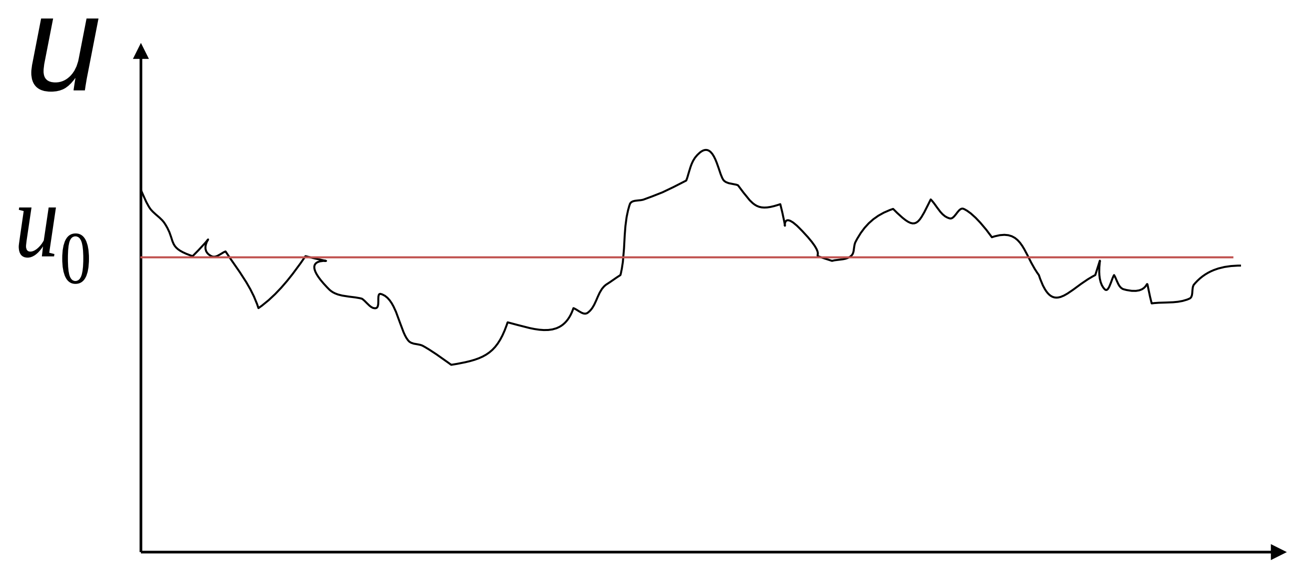
Synaptic current pulses

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\{ \sum_{k,f} q_e \delta(t - t_k^f) - \sum_{k',f'} q_i \delta(t - t_{k'}^{f'}) \}$$

EPSC

IPSC

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\, I^{mean}(t) + \xi(t)$$



Langevin equation,
Ornstein Uhlenbeck process

Week 13 part 2 – Membrane Potential Densities

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Biological Modeling of Neural Networks:

Week 13 – Membrane potential densities and Fokker-Planck

Wulfram Gerstner

EPFL, Lausanne, Switzerland

13.1 Review: Integrate-and-fire
- stochastic spike arrival

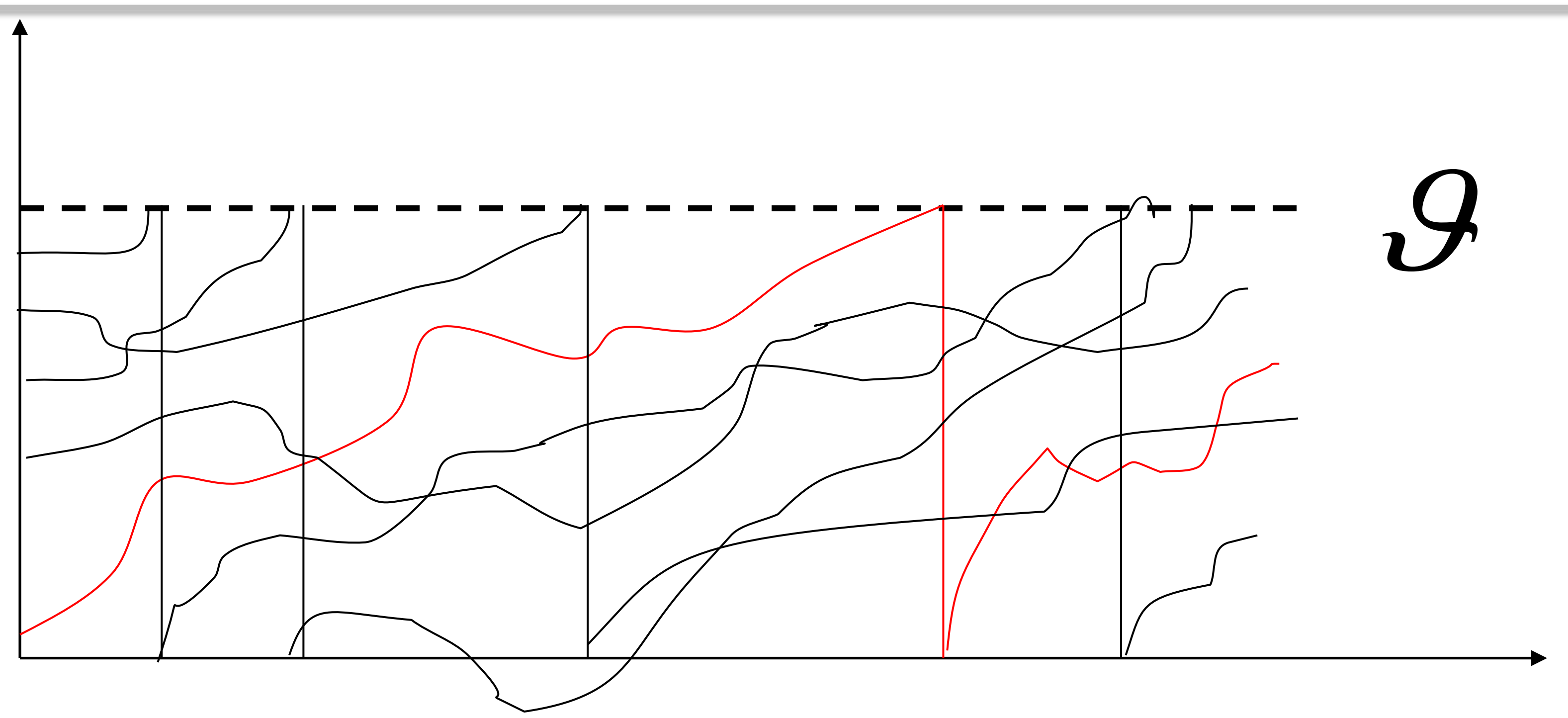
13.2 Density of membrane potential
-

13.3 Flux
-
-

13.4. Fokker –Planck Equation
-
-

13.5. Threshold and reset
-

Week 13-part 2: membrane potential density



Blackboard: density of potentials?

For any arbitrary neuron in the population

$$\tau \frac{d}{dt}u = -u + R(\sum_{k,f} q_e \delta(t - t_k^f) - \sum_{k',f'} q_i \delta(t - t_{k'}^{f'}))$$

EPSC

IPSC

$$\frac{d}{dt}u = -\frac{u}{\tau} + \sum_{k,f} \frac{q_e}{C} \delta(t - t_k^f) + I^{ext}(t)$$

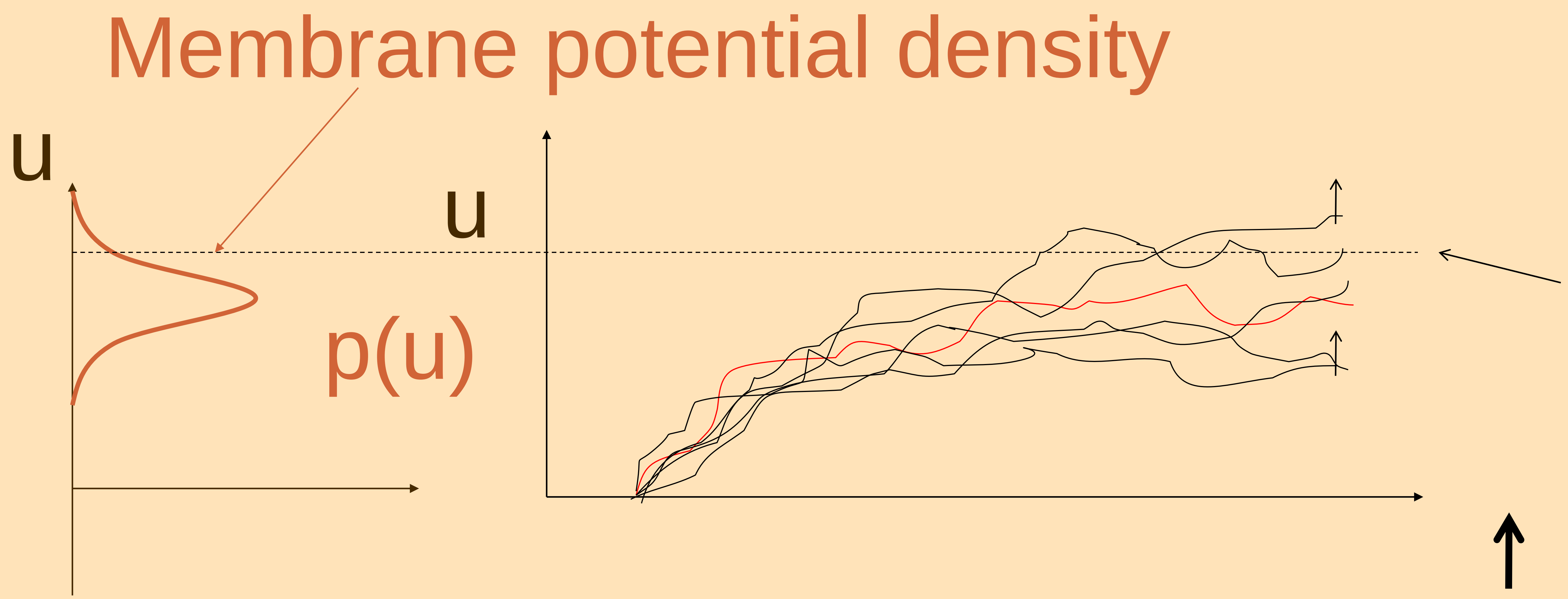
excitatory input spikes

external current input

Week 13-part 2: continuity equation

$$\frac{d}{dt}p(u,t) = -\frac{d}{du}J(u,t)$$

Exercise 1: flux caused by stochastic spike arrival



Next lecture: 10h15

Reference level u_0

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\{ \sum_f q_e \delta(t - t^f) \}$$

a) Jump at time t

What is the flux across u_0?

b) spike arrival rate ν

c) spike arrival rate $\sum_k \nu_k$

Week 13 – Membrane Potential Densities and Fokker-Planck Equation

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Biological Modeling of Neural Networks:

Week 13 – Membrane potential densities and Fokker-Planck

Wulfram Gerstner

EPFL, Lausanne, Switzerland

13.1 Review: Integrate-and-fire

- stochastic spike arrival

13.2 Density of membrane potential

- continuity equation

13.3 Flux

- jump flux
- drift flux

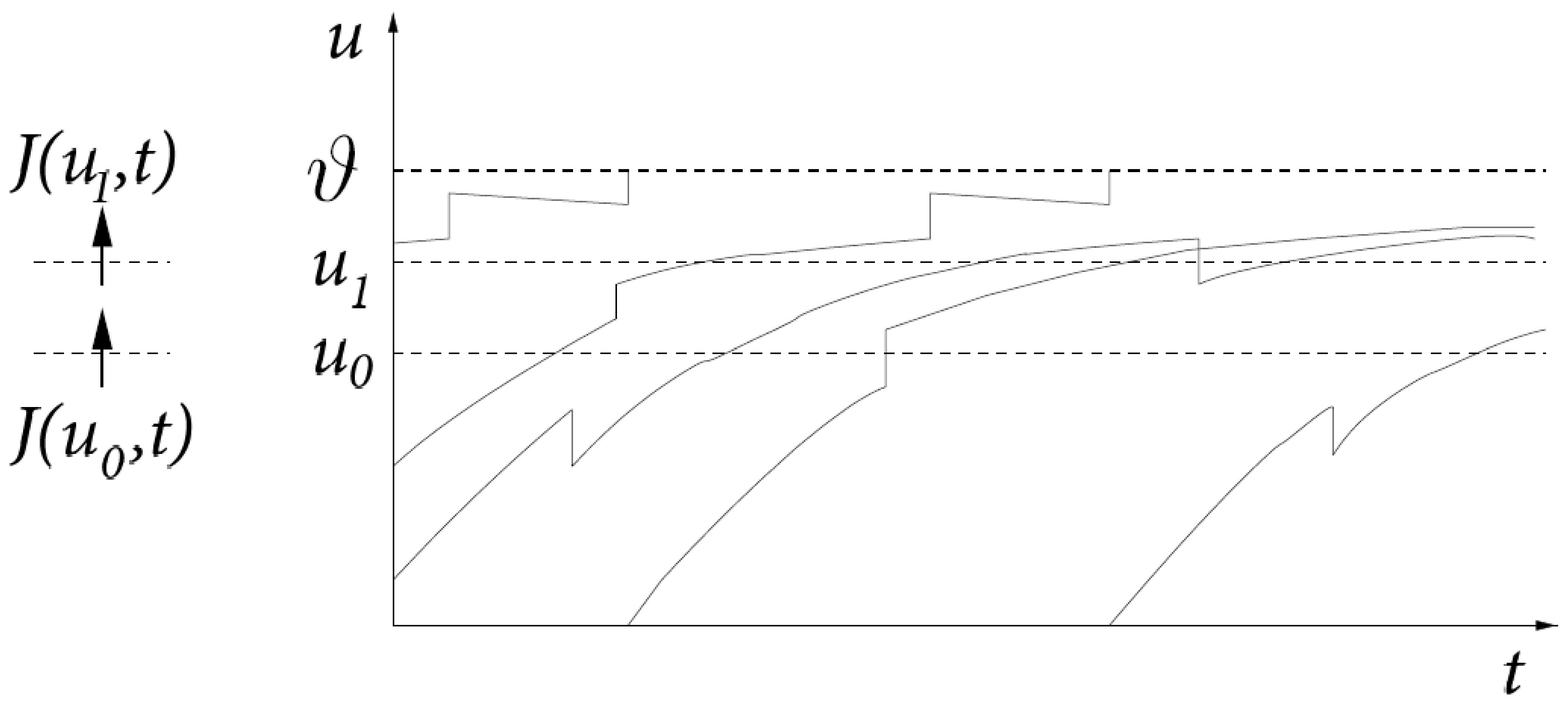
13.4. Fokker –Planck Equation

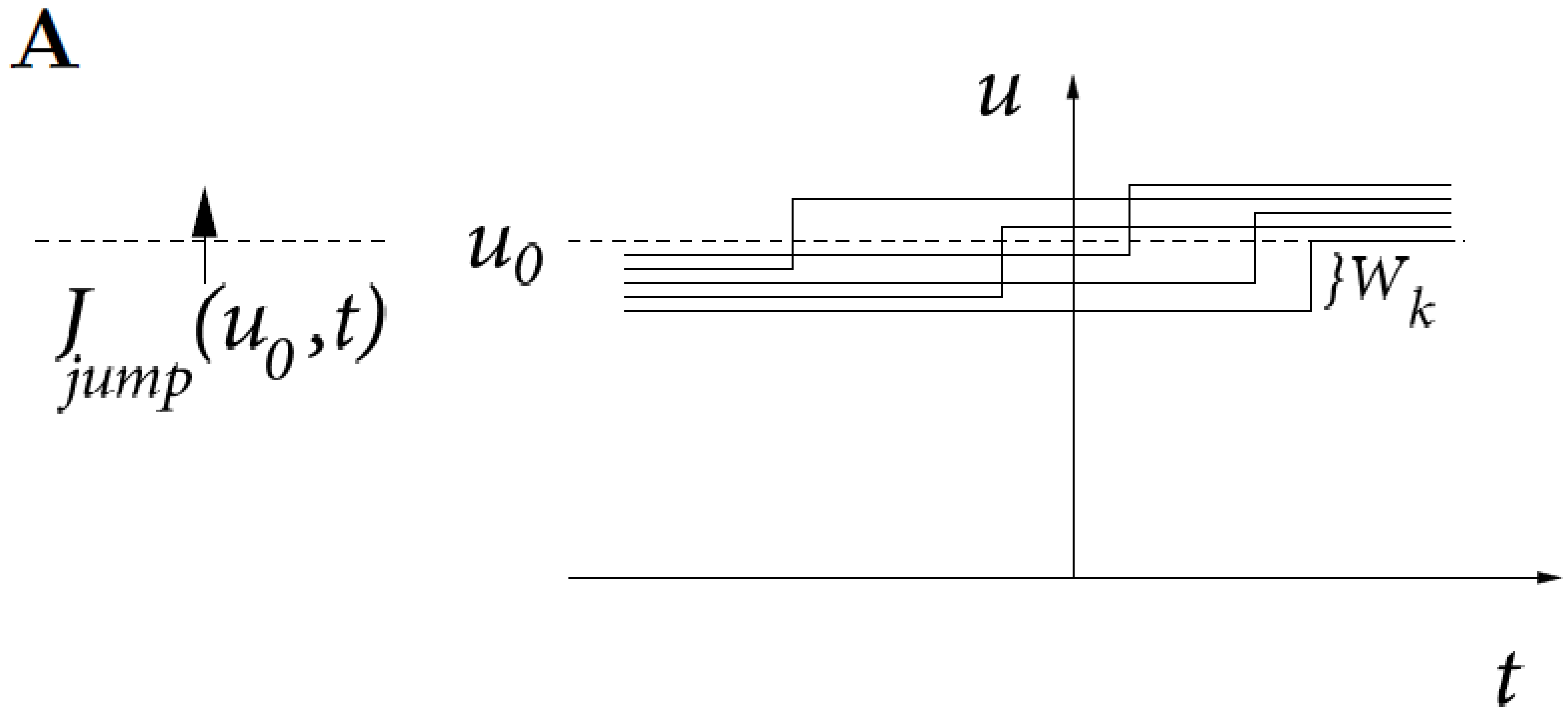
-

13.5. Threshold and reset

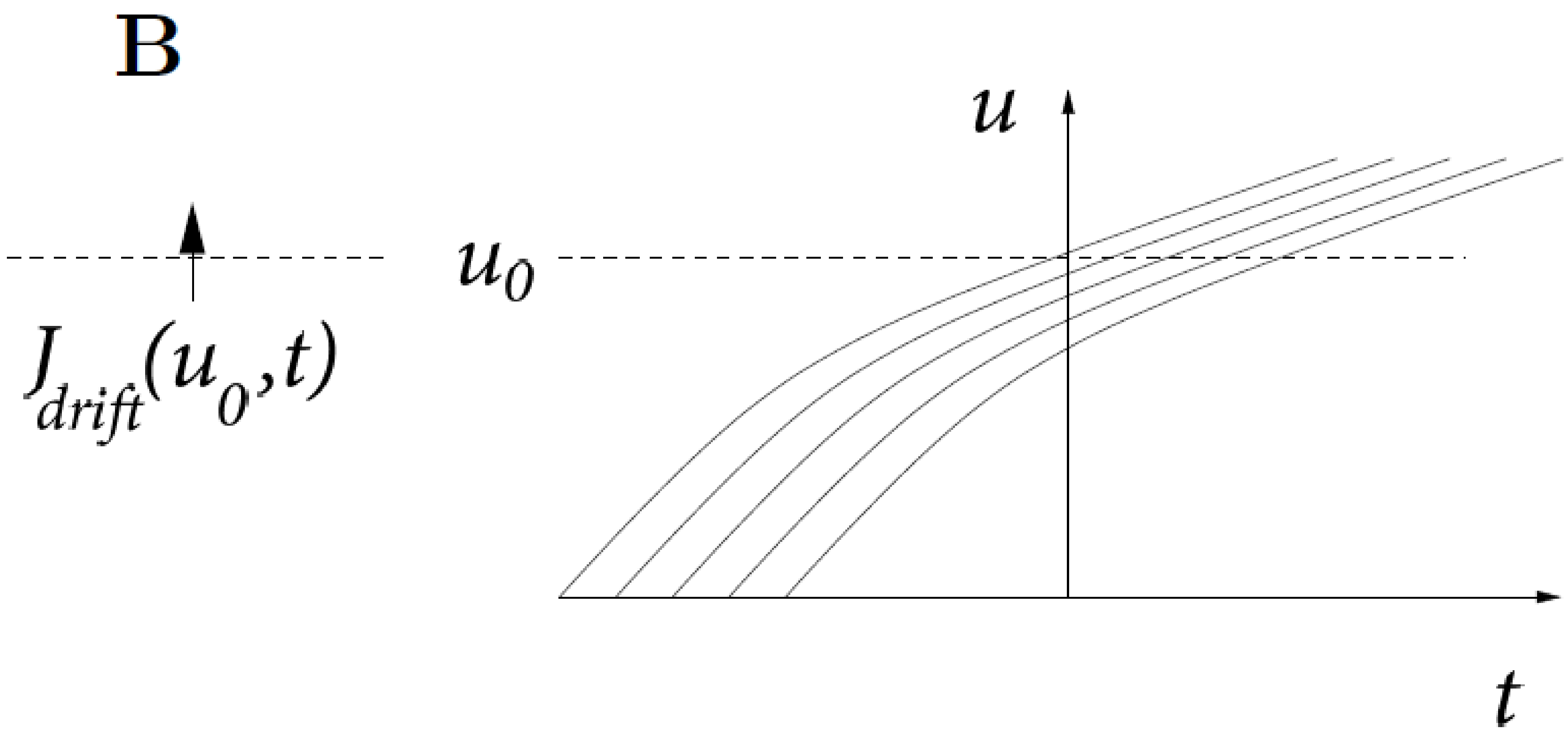
-

Week 13-part 2: membrane potential density: flux by jumps





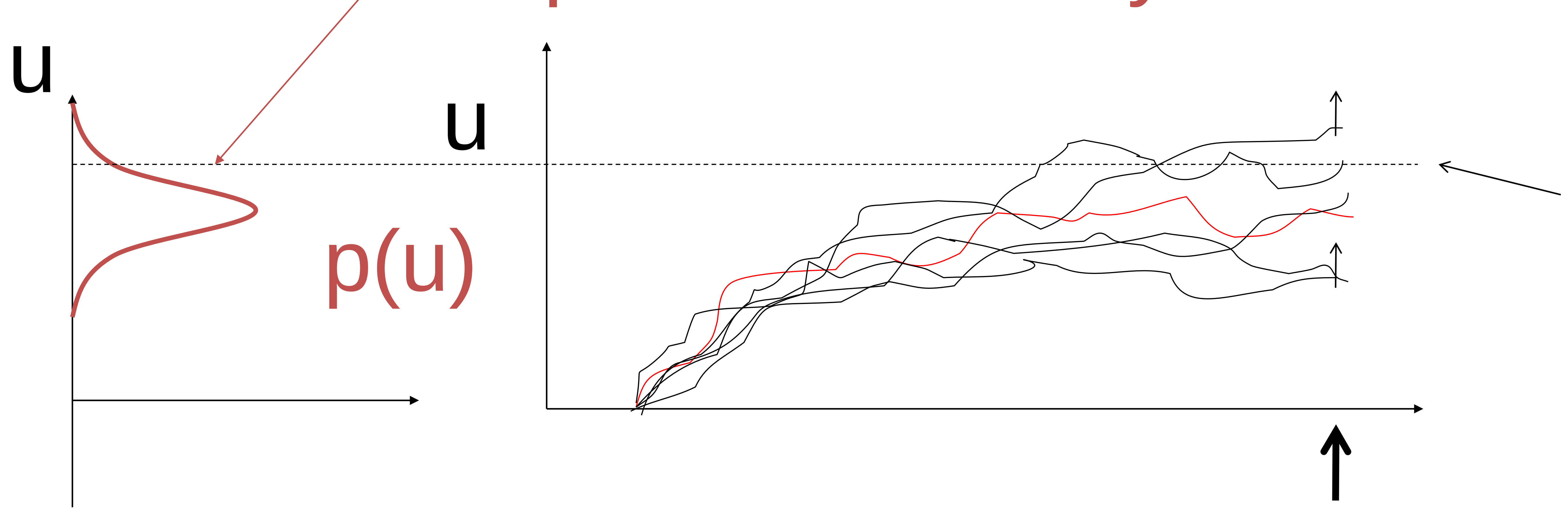
Week 13-part 2: membrane potential density: flux by drift



flux – two possibilities

$$\tau \frac{d}{dt}u = -(u - u_{eq}) + R\{I^{ext(t)} + \sum_f q_e \delta(t - t^f)\}$$

Membrane potential density



What is the flux

across u_0?

Reference level u_0

Jumps caused at

spike arrival rate ν

a) flux caused by jumps due to stochastic spike arrival

b) flux caused by systematic drift

Blackboard:
Slope and density of potentials

Week 13 – Membrane Potential Densities and Fokker-Planck Equation

Biological Modeling of Neural Networks:

Week 13 – Membrane potential densities and Fokker-Planck

Wulfram Gerstner

EPFL, Lausanne, Switzerland

13.1 Review: Integrate-and-fire
- stochastic spike arrival

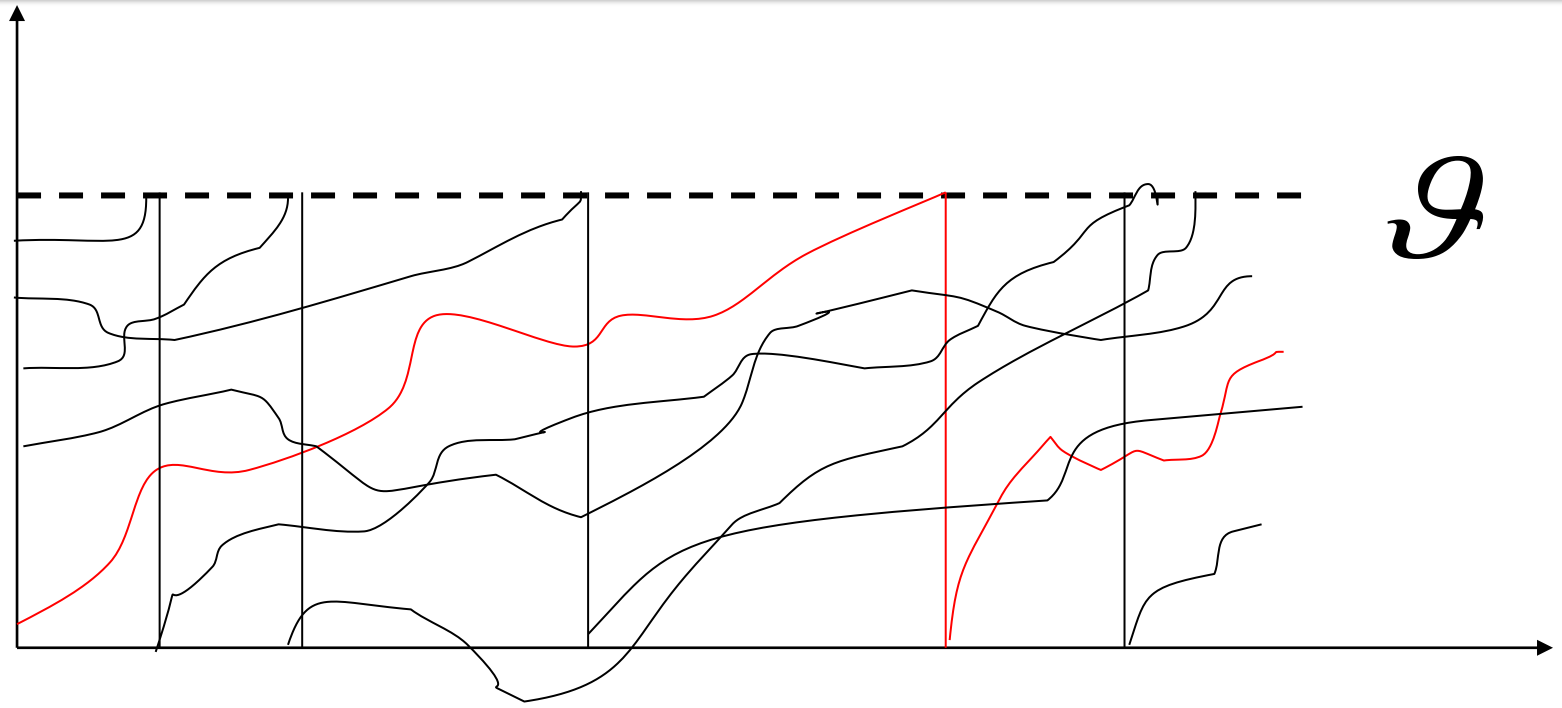
13.2 Density of membrane potential
-

13.3 Flux
- continuity equation

13.4. Fokker –Planck Equation
- source and sink
-

13.5. Threshold and reset
-

Week 13-part 4: from continuity equation to Fokker-Planck



Blackboard: Derive Fokker-Planck equation

For any arbitrary neuron in the population

$$\frac{d}{dt}u = -\frac{u}{\tau} + \sum_{k,f} \frac{q_e}{C}\delta(t - t_k^f) - \sum_{k',f'} \frac{q_i}{C}\delta(t - t_{k'}^{f'}) + \frac{1}{C} I^{ext}(t)$$

EPSC IPSC external input

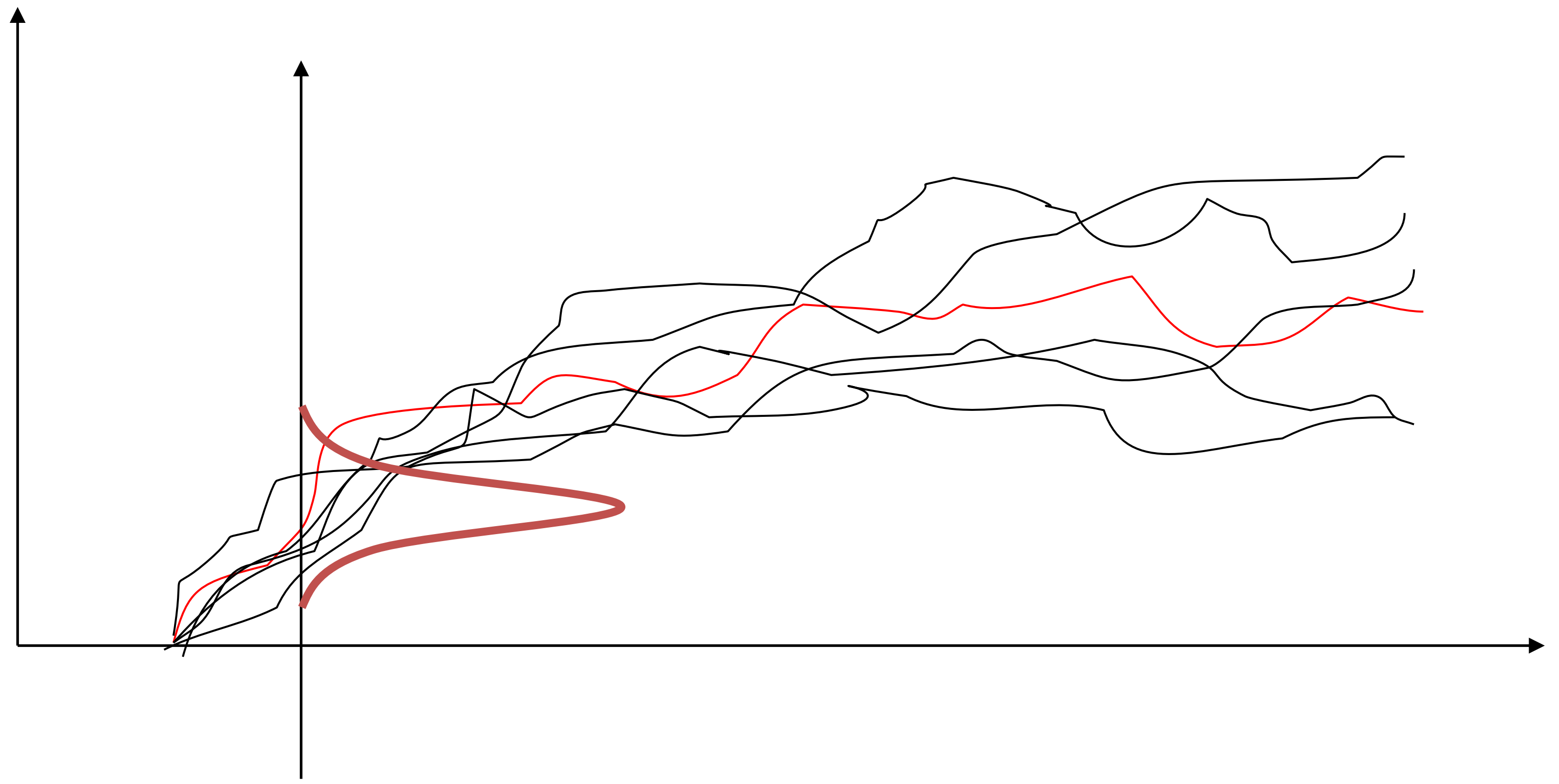
Continuity equation:

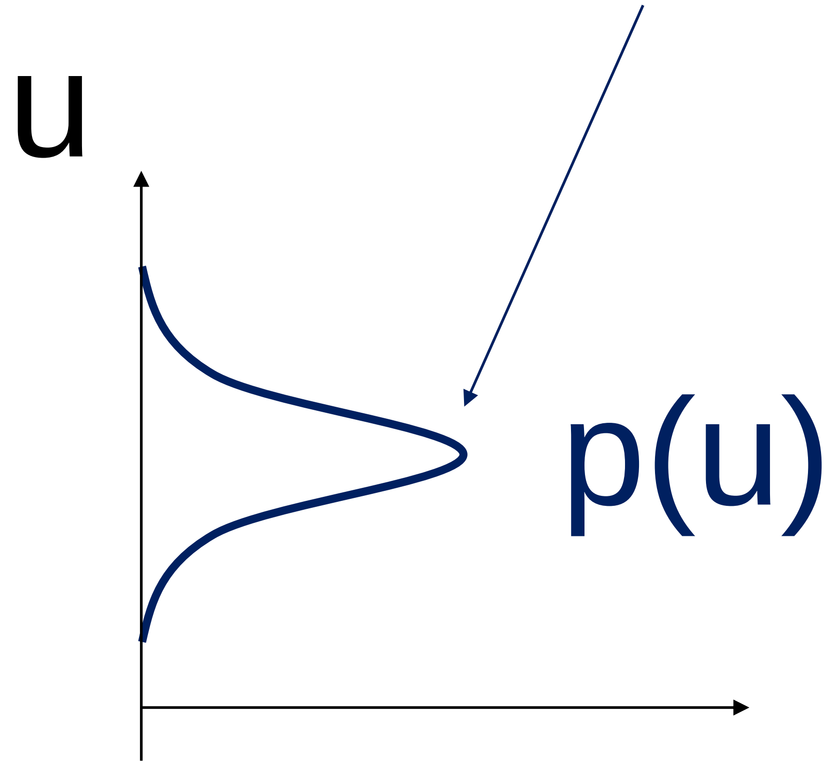
$$\frac{\partial}{\partial t} p(u,t) = -\frac{\partial}{\partial u} J(u,t)$$

Flux:
- jump (spike arrival)
- drift (slope of trajectory)

Week 13-part 4: Fokker-Planck equation

Membrane potential density





Fokker-Planck

$$\tau \cdot \frac{\partial}{\partial t} p(u,t) = -\frac{\partial}{\partial u}[\gamma(u)\, p(u,t)] + \sigma^2 \frac{\partial^2}{\partial u^2} p(u,t)$$

drift

$$\gamma(u) = -u + \tau \sum_k \nu_k w_k$$

spike arrival rate

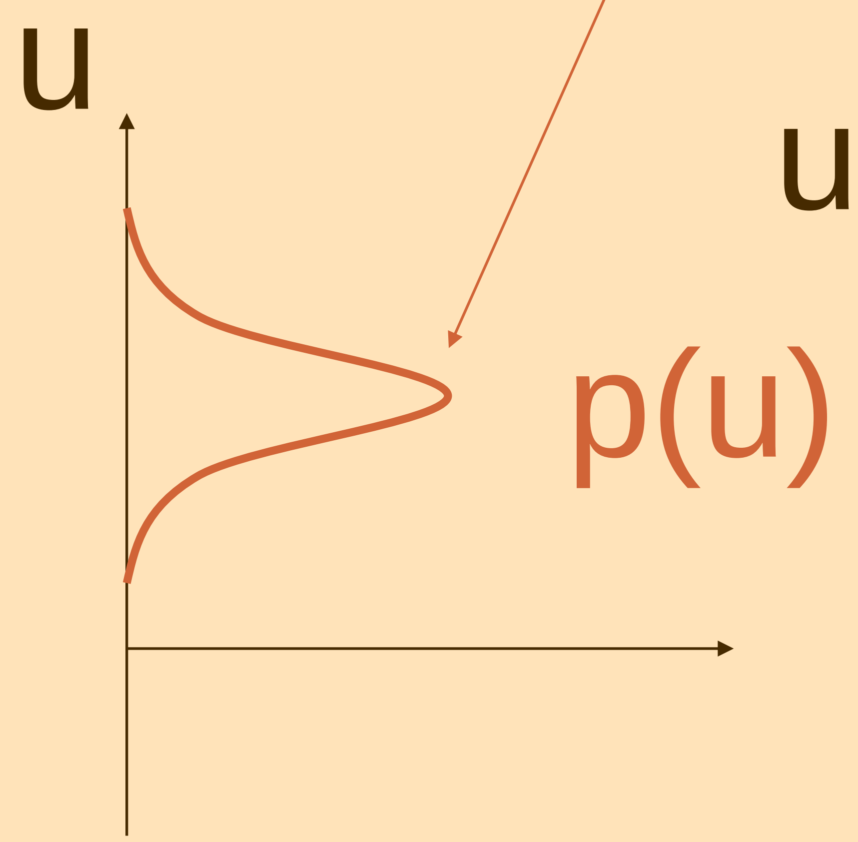
diffusion

$$\sigma^2 = \frac{1}{2} \tau \sum_k \nu_k w_k^2$$

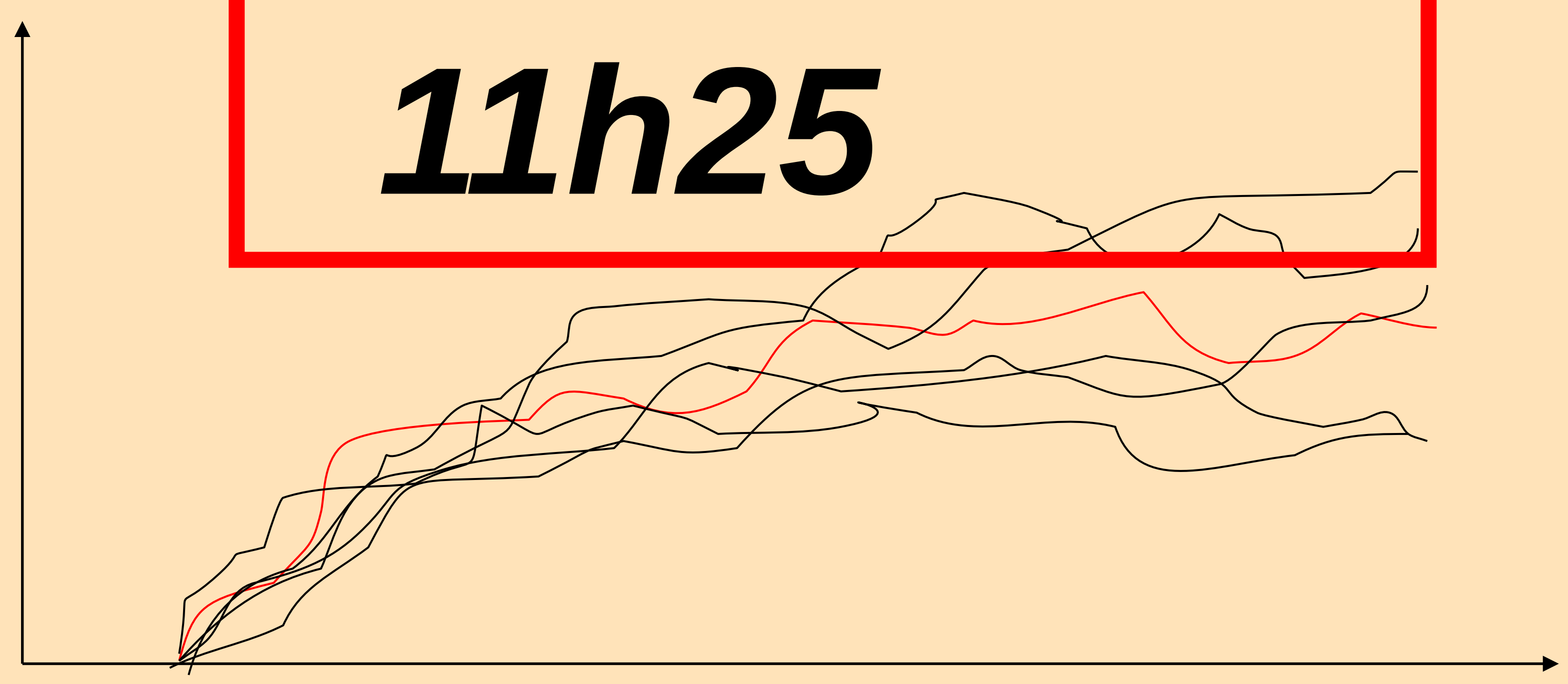
Exercise 2: solution of free Fokker-Planck equation

Membrane potential density: Gaussian

Next lecture:
11h25



constant input rates
no threshold



Fokker-Planck

$$\tau \cdot \frac{\partial}{\partial t} p(u,t) = -\frac{\partial}{\partial u}[\gamma(u)\, p(u,t)] + \sigma^2 \frac{\partial^2}{\partial u^2} p(u,t)$$

drift

$$\gamma(u) = -u + \tau \sum_k \nu_k w_k + RI(t)$$

spike arrival rate

diffusion

$$\sigma^2 = \frac{1}{2}\tau \sum_k \nu_k w_k^2$$

Week 13 – Membrane Potential Densities and Fokker-Planck Equation

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Biological Modeling of Neural Networks:

Week 13 – Membrane potential densities and Fokker-Planck

Wulfram Gerstner

EPFL, Lausanne, Switzerland

13.1 Review: Integrate-and-fire
- stochastic spike arrival

13.2 Density of membrane potential
-

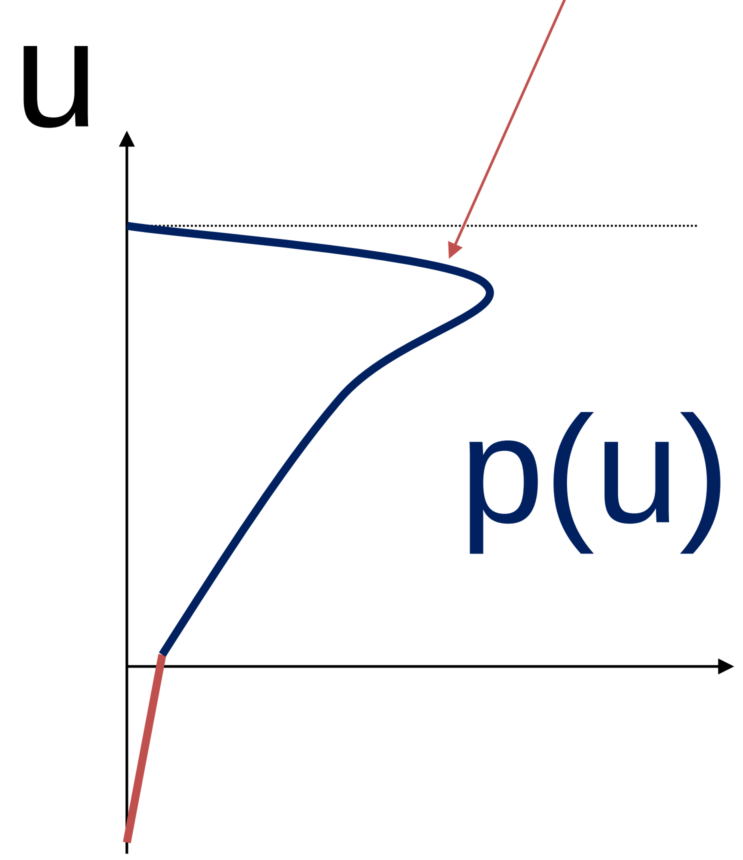
13.3 Flux
- continuity equation

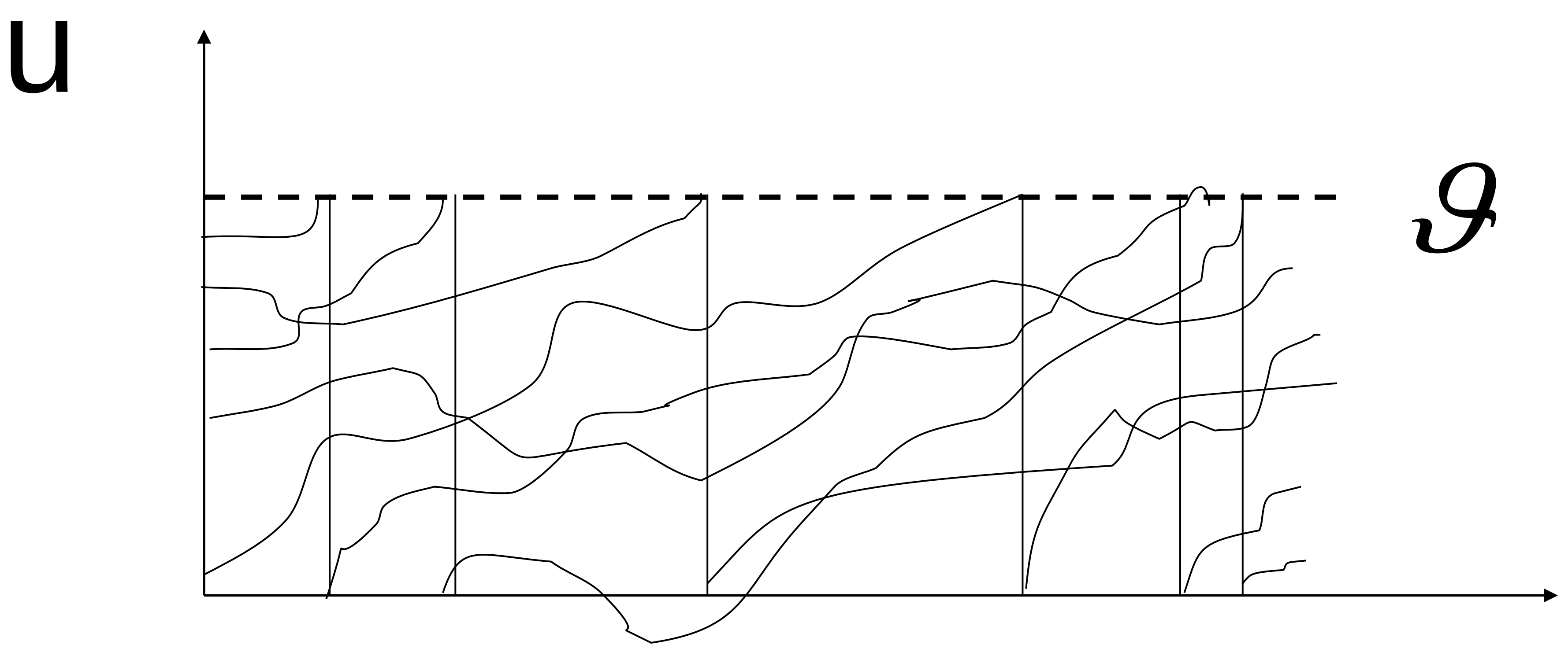
13.4. Fokker –Planck Equation
-
-

13.5. Threshold and reset
-

Week 13-part 5: Threshold and reset (sink and source terms)

Membrane potential density





blackboard

Fokker-Planck

$$\tau \cdot \frac{\partial}{\partial t} p(u,t) = -\frac{\partial}{\partial u}[\gamma(u)\, p(u,t)] + \sigma^2 \frac{\partial^2}{\partial u^2} p(u,t) + \tau A(t)\, \delta(u - u_{reset})$$

drift

$$\gamma(u) = -u + \tau \sum_k \nu_k w_k + RI$$

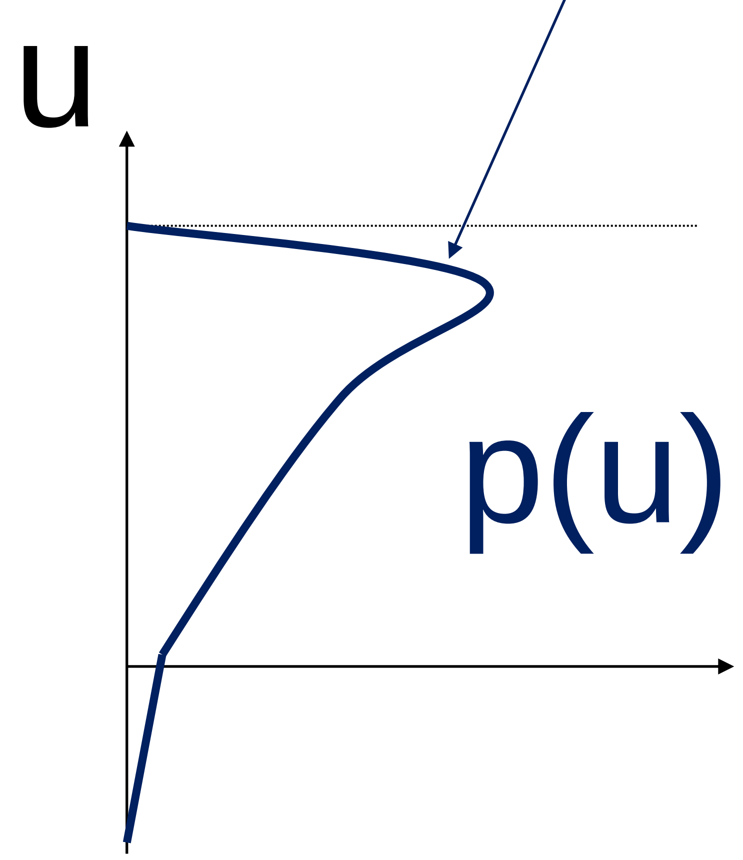
diffusion

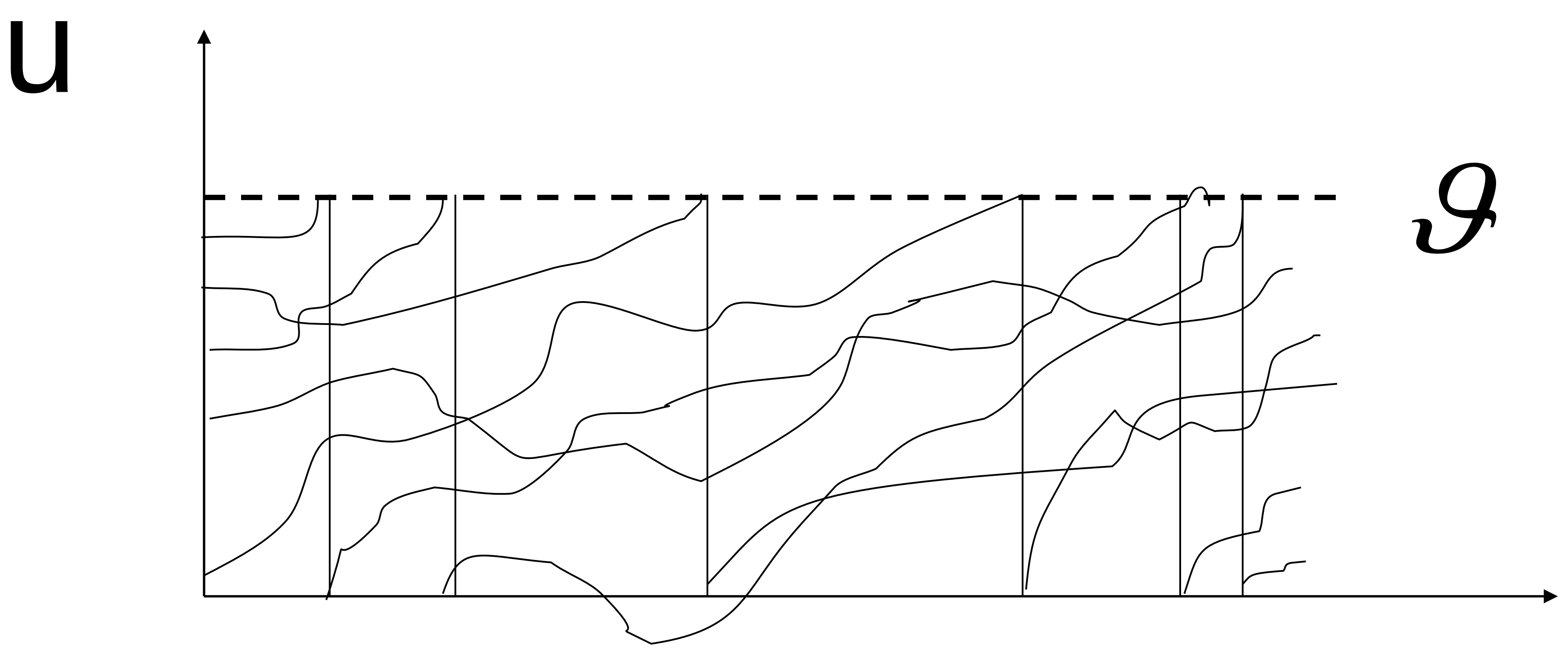
$$\sigma^2 = \frac{1}{2} \tau \sum_k \nu_k w_k^2$$

spike arrival rate

Week 13-part 5: population firing rate A(t)

Membrane potential density





blackboard

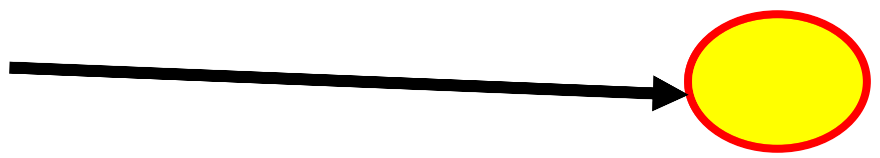
Population Firing rate *A(t):* flux at threshold

Week 13-part 5: population firing rate A(t) = single neuron rate

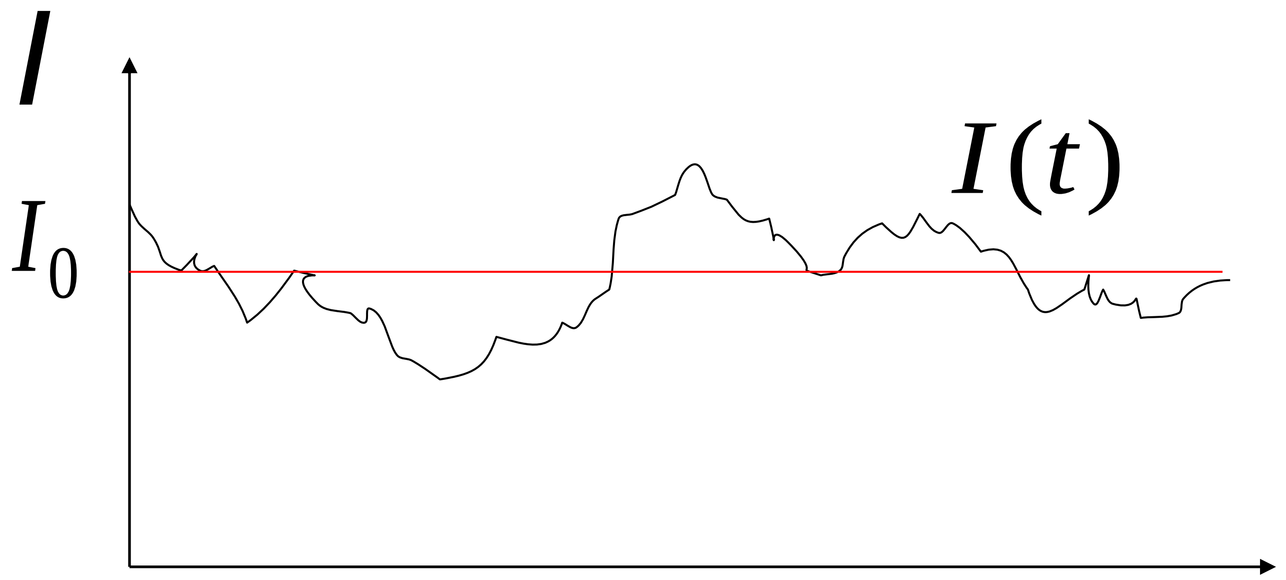
Synaptic current pulses

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\{ \sum_{k,f} q_e \delta(t - t_k^f) - \sum_{k',f'} q_i \delta(t - t_{k'}^{f'}) \}$$

EPSC IPSC



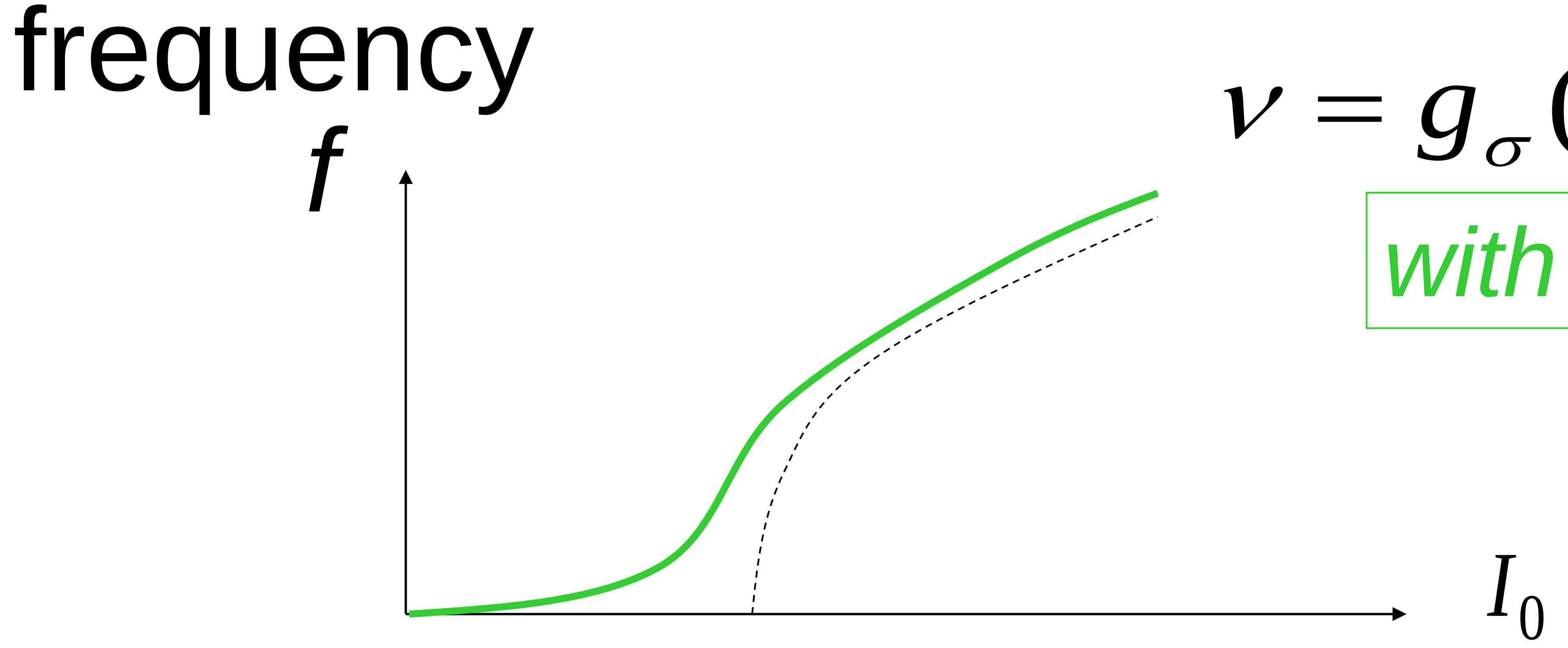
$$\tau \frac{d}{dt} u = -(u - u_{eq}) \quad + R I^{mean}(t) + \quad \xi(t)$$



$$\tau \frac{d}{dt} u = -(u - u_{eq}) \quad + R I(t)$$

$$I(t) = [I_o + I_{noise}]$$

effective noise current



$$\nu = g_\sigma(I)$$

with noise

Week 13-part 5: membrane potential density

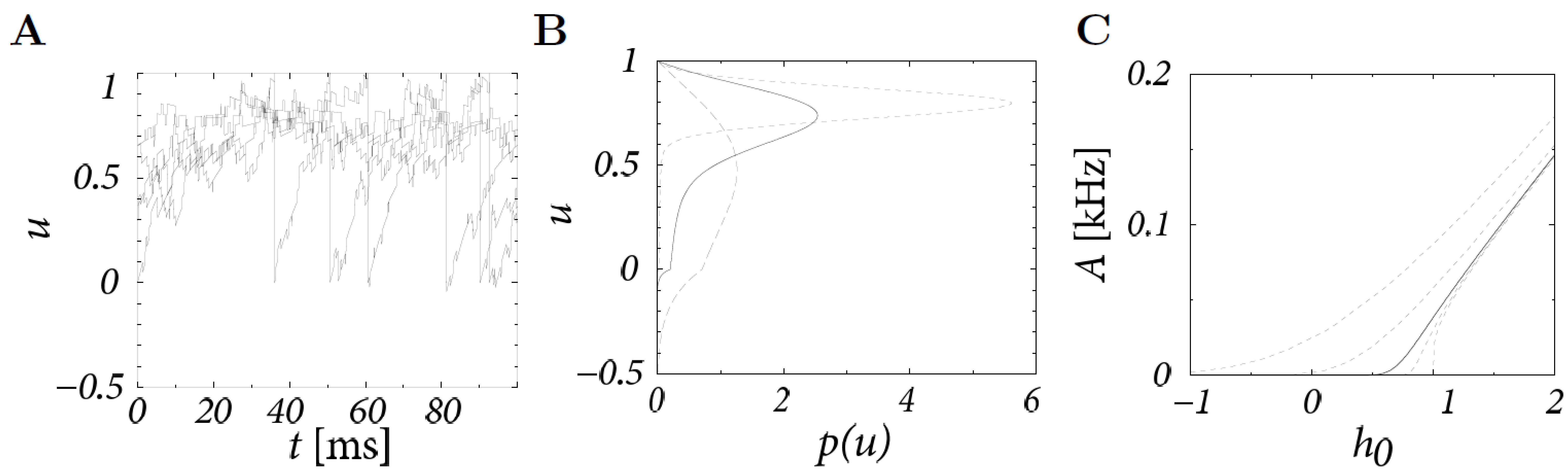
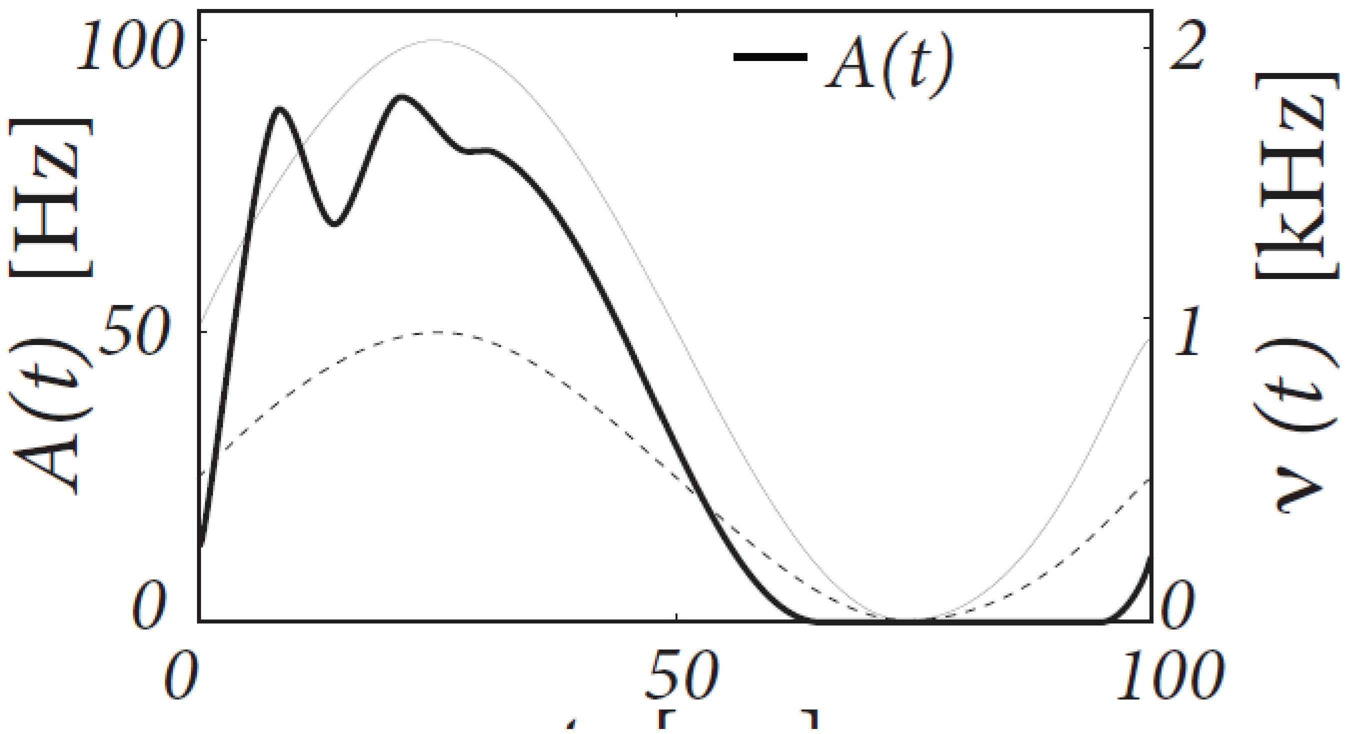
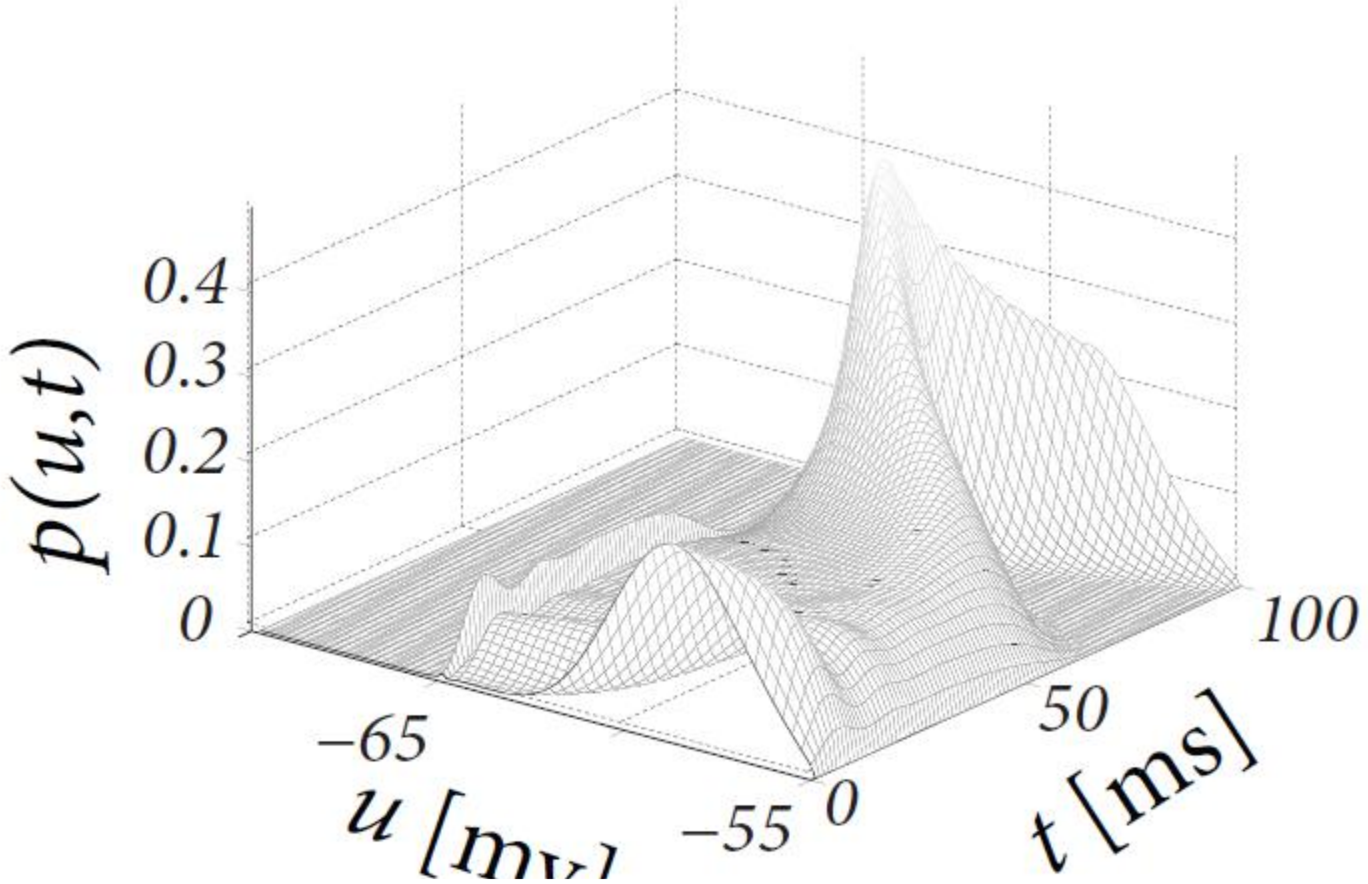
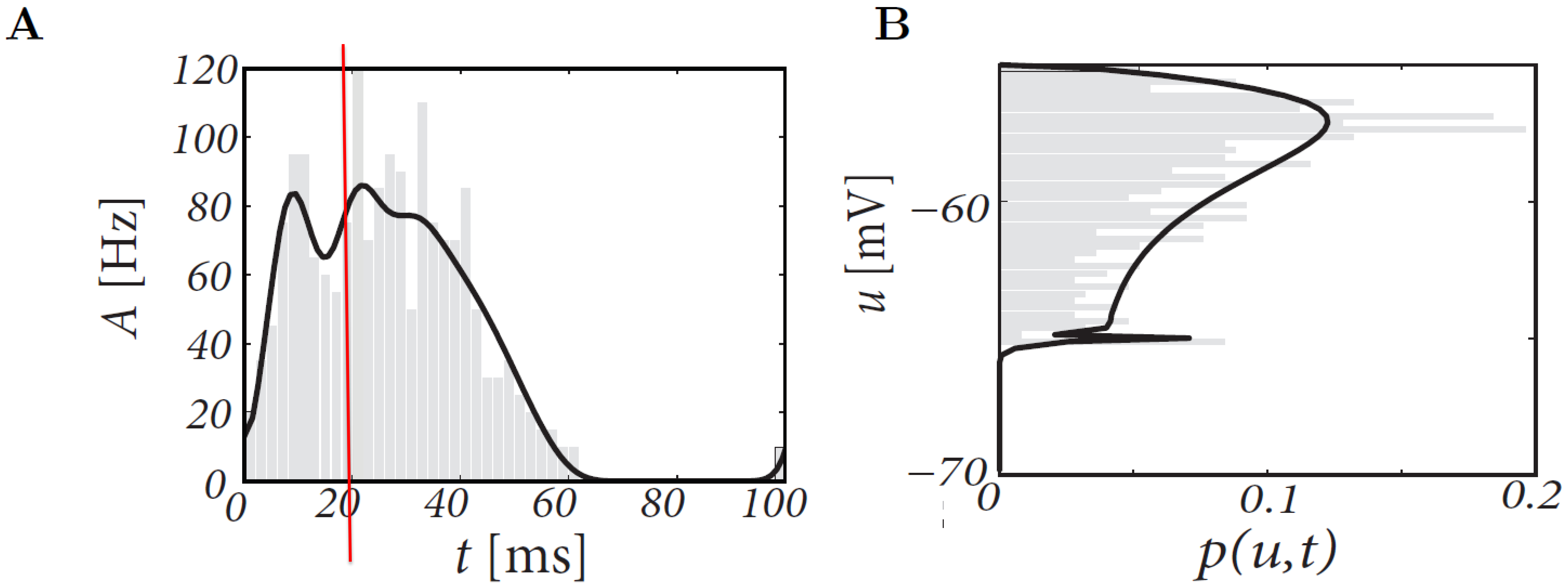


Fig. 13.5: **A**. Membrane potential trajectories of 5 neurons ($R = 1$ and $\tau_m = 10\,\text{ms}$) driven by a constant background current $I_0 = 0.8$ and stochastic background input with $\nu_+ = \nu_- = 0.8\,\text{kHz}$ and $w_\pm = \pm 0.05$. These parameters correspond to $h_0 = 0.8$ and $\sigma = 0.2$ in the diffusive noise model. **B**. Stationary membrane potential distribution in the diffusion limit for $\sigma = 0.2$ (solid line), $\sigma = 0.1$ (short-dashed line), and $\sigma = 0.5$ (long-dashed line). (Threshold $\vartheta = 1$). **C**. Mean activity of a population of integrate-and-fire

Week 13-part 5: population activity, time-dependent



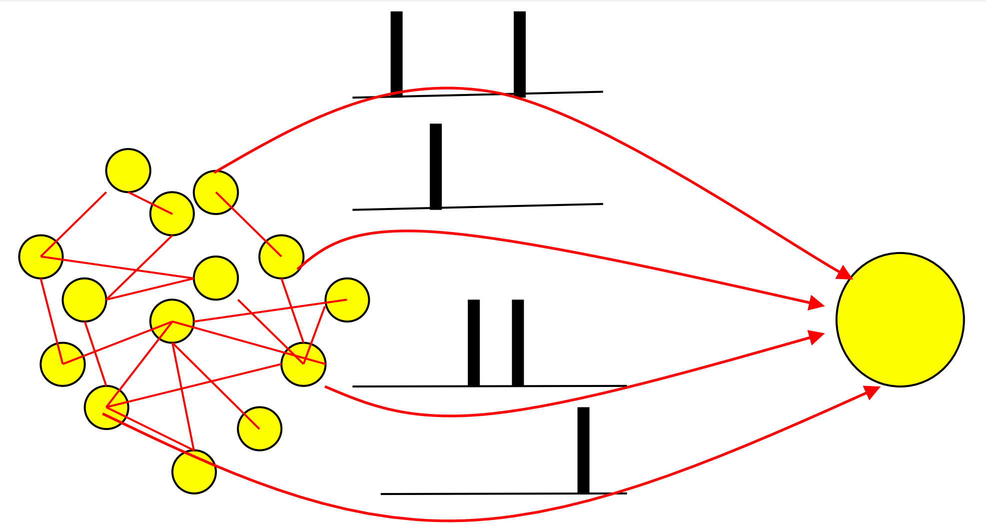


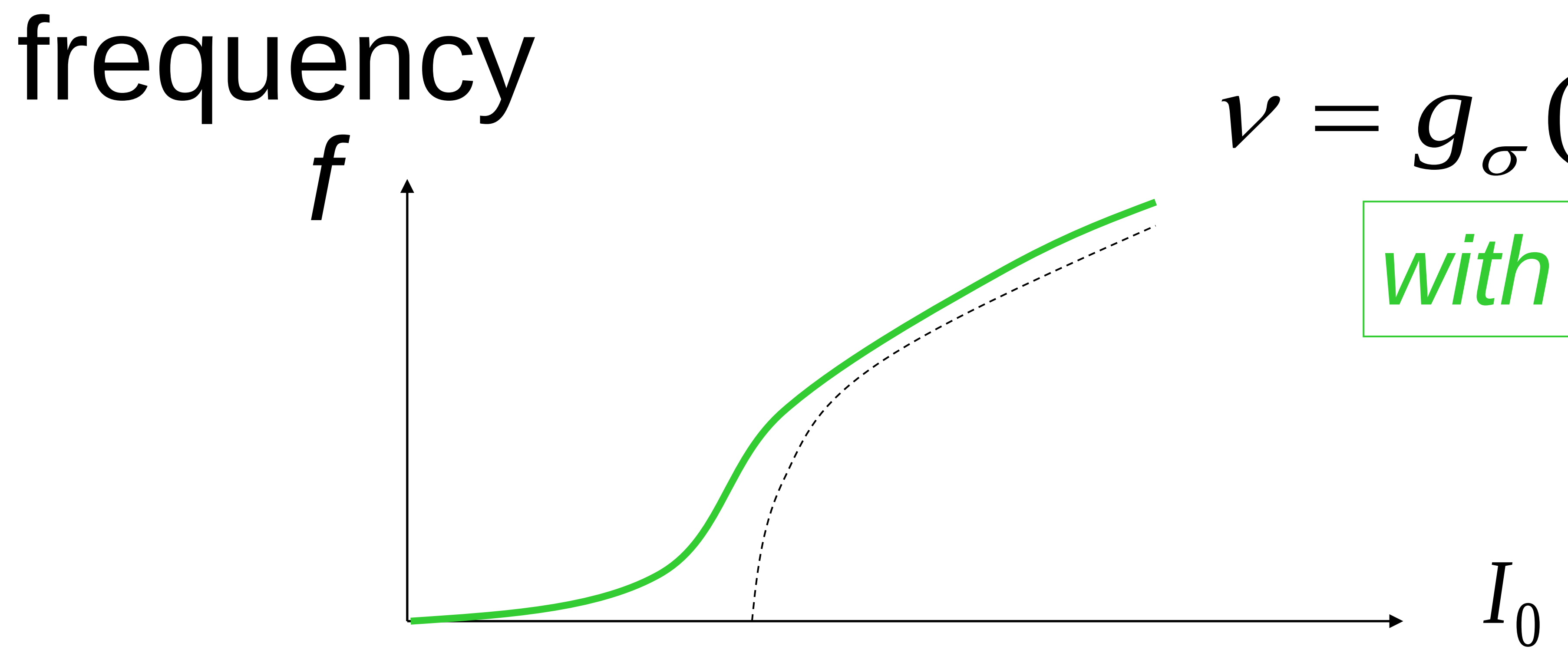


Nykamp+Tranchina, 2000

Fig. 13.4: Comparison of theory and simulation. **A**. Population firing rate $A(t)$ as a function of time in a simulation of 1 000 neurons (histogram bars) compared to the prediction

Week 13-part 5: network states





mean *I(T)* depends on state

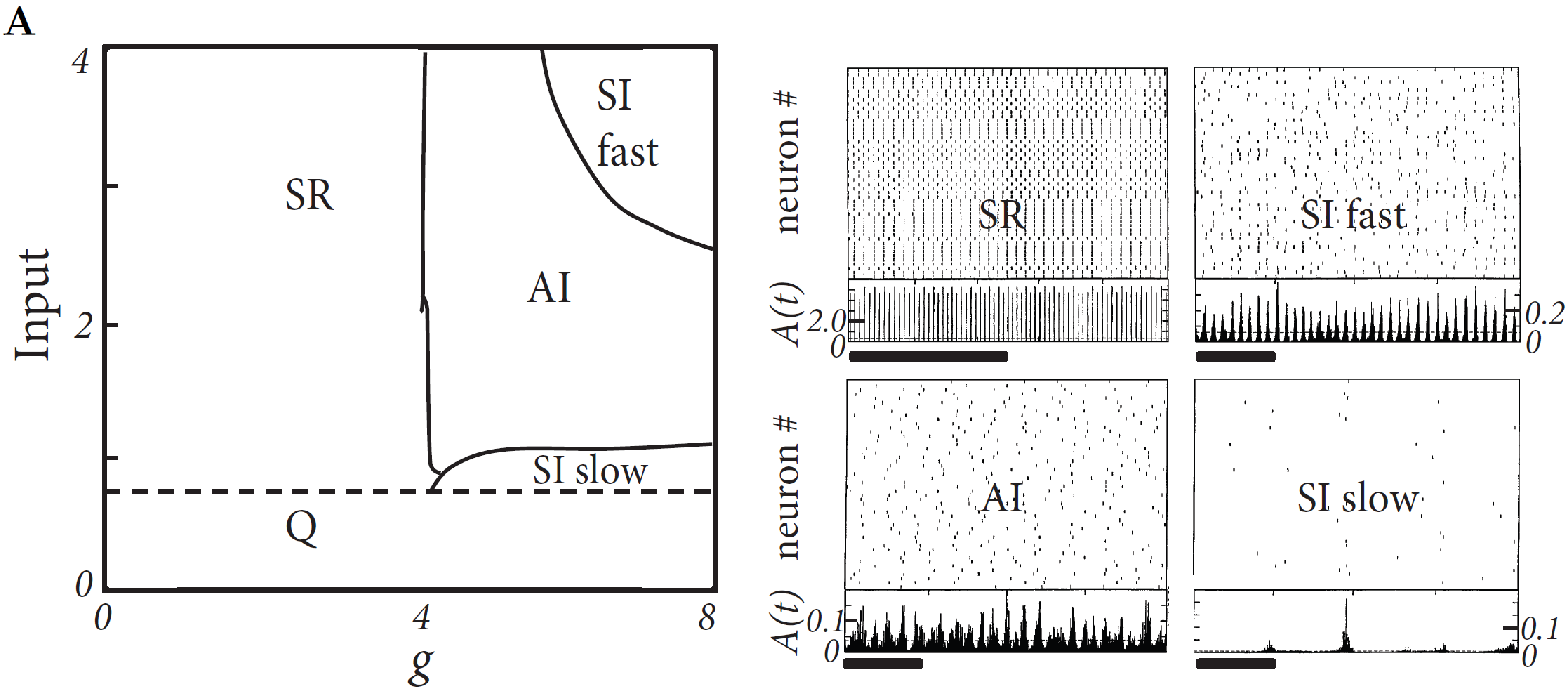
Variance/noise depends on state

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\{ \sum_{k,f} q_e \delta(t - t_k^f) - \sum_{k',f'} q_i \delta(t - t_{k'}^{f'}) \}$$

EPSC IPSC

$$\tau \frac{d}{dt} u = -(u - u_{eq}) + R\, I^{mean}(t) + \sigma \xi(t)$$

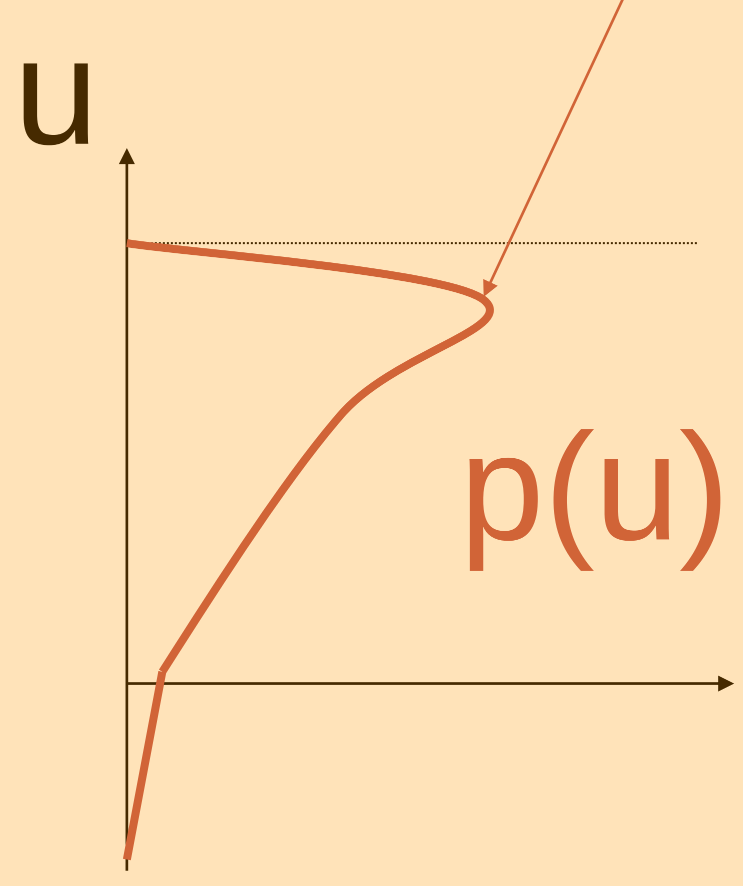
Week 13-part 5: network states



Brunel 2000

Exercise 3: Diffusive noise + Threshold

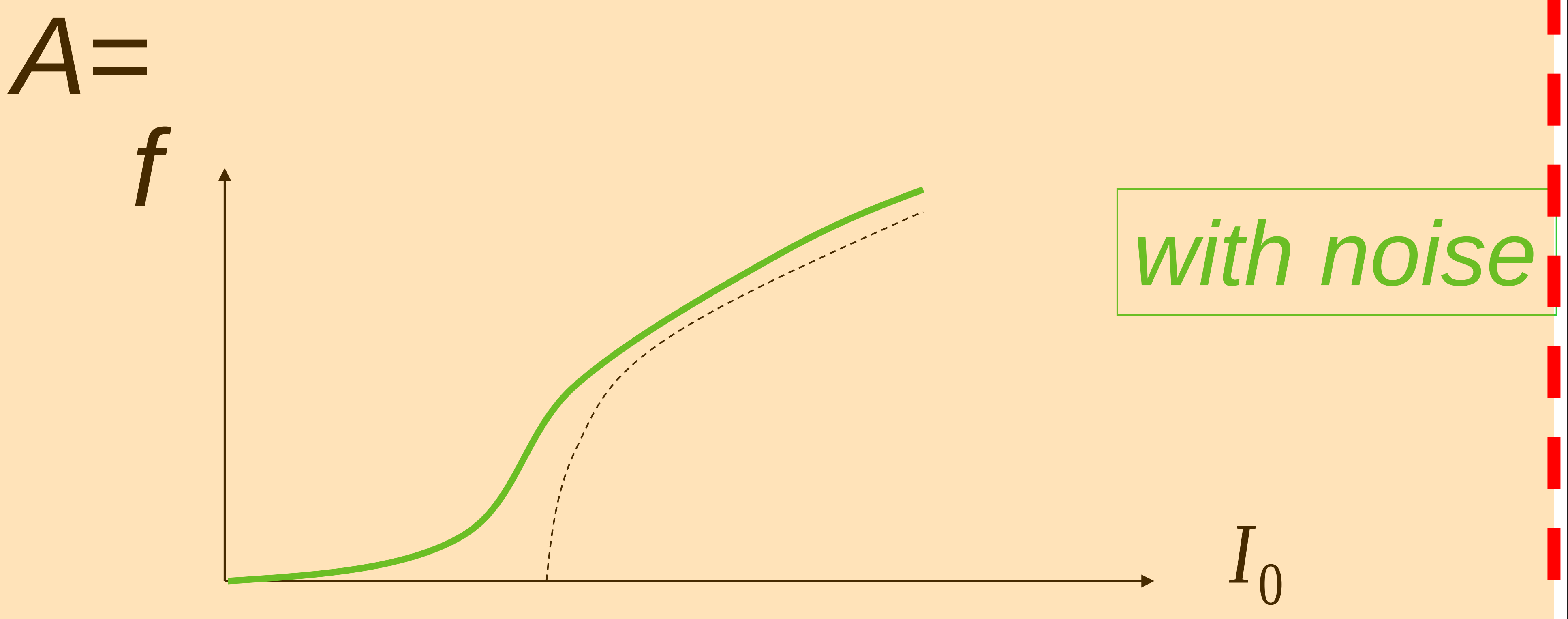
Membrane potential density



u

Fokker-Planck

$$\tau \cdot \frac{\partial}{\partial t} p(u,t) = - \frac{\partial}{\partial u}[\gamma(u)\, p(u,t)] + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial u^2} p(u,t) + source$$



Miniproject: 12h00

- Calculate distribution *p(u)*
- Determine population firing rate *A*

THE END

Sign up for:

- miniproject fraud detection interview